NPTEL ONLINE CERTIFICATION COURSES

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Finite Element Method

Week 8: Finite element formulation for 2D scalar field problems

Lecture 37: Weak Form

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Problem

A model problem

Find the solution $u(x,y)$ of the second order PDE

$$\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)-a_{00}u+f=0,$$

for given a_{11}, a_{12}, a_{21}, a_{22}, a_{00}, f and specified boundary conditions

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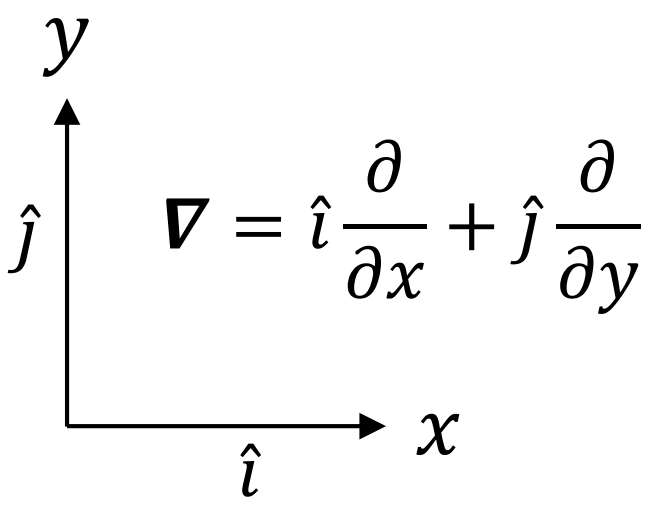
Specific case ($a_{11} = a_{22} = k$, $a_{00} = a_{12} = a_{21} = 0$)

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial u}{\partial x}\right)+f=0$$

$$\boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}u)=-f$$ Poisson equation



$$\nabla \cdot (k\nabla u) = -f \text{ in } \Omega$$

Governing equations

$$u = \hat{u} \text{ on } \Gamma_u$$

Essential BC

$$k\,\partial u/\partial n + \beta(u - u_\infty) = q \text{ on} \Gamma_q$$

Natural BC

Field of applications	Primary variable *u*	Material constant *k*	Source variable *f*	Secondary variable *q*, $\partial u/\partial x$, $\partial u/\partial y$
Heat Transfer	Temperature T	Conductivity *k*	Heat source *g*	Heat flow due to conduction $k\partial T/\partial n$ Convection $h(T - T_\infty)$
Irrotational flow of an ideal fluid	Stream function Ψ Velocity potential Φ	Density (ρ)	Mass production σ	Velocities $\partial\Psi/\partial x = -v$ $\partial\Psi/\partial y = u$ $\partial\Phi/\partial x = u$ $\partial\Phi/\partial y = v$
Ground water flow	Piezometric head Φ	Permeability *K*	Recharge *f* (pumping, $-f$)	Seepage $q = k\partial\Phi/\partial n$ Velocities $k\partial\Phi/\partial x = -u$ $k\partial\Phi/\partial y = -v$
Torsion of cylindrical members with constant cross-section	Stress function Ψ	$k = 1$ G = shear modulus	$f = 2$ θ = angle of twist per unit length	$G\theta\partial\Psi/\partial x = -\sigma_{yz}$ $G\theta\partial\Psi/\partial y = \sigma_{xz}$
Electrostatics	Scalar potential Φ	Dielectric constant (ε)	Charge density ρ	Displacement flux D_n
Magnetostatics	Magnetic potential Φ	Permeability (μ)	Charge density ρ	Displacement flux B_n
Transverse deflection of elastic membrane	Transverse deflection *u*	Tension in member *T*	Transversely distributed load	Normal force *q*

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Weak Form

$$\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)-a_{00}u+f=0, \qquad \text{in } \Omega$$

$$\int_{\Omega} w\left[\frac{\partial}{\partial x}\underbrace{\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)}_{F_1}+\frac{\partial}{\partial y}\underbrace{\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)}_{F_2}-a_{00}u+f\right]dydx=0,$$

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$$\int_{\Omega} w\left[\frac{\partial}{\partial x}(F_1)+\frac{\partial}{\partial y}(F_2)-a_{00}u+f\right]dydx=0$$

Weak Form

$$\int_\Omega w\left[\frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) - a_{00}u + f\right]dxdy = 0,$$

(1) (2) (3) (4)

$$F_1 = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}$$

$$F_2 = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}$$

① $$\int_\Omega w\left[\frac{\partial}{\partial x}(F_1)\right]dxdy \Rightarrow \int_\Omega \left[\frac{\partial}{\partial x}(wF_1) - \frac{\partial w}{\partial x}F_1\right]dxdy$$

Divergence theorem

$$\int_\Omega \frac{\partial}{\partial x}(wF_1)dxdy = \oint_\Gamma wF_1 n_x ds$$

② $$\int_\Omega w\left[\frac{\partial}{\partial y}(F_2)\right]dxdy \Rightarrow \int_\Omega \left[\frac{\partial}{\partial y}(wF_2) - \frac{\partial w}{\partial y}F_2\right]dxdy$$

Divergence theorem

$$\int_\Omega \frac{\partial}{\partial y}(wF_2)dxdy = \oint_\Gamma wF_2 n_y ds$$

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Weak Form

$$\int_{\Omega} w\left[\frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) - a_{00}u + f\right] dxdy = 0,$$

(1) (2) (3) (4)

$$F_1 = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}$$

$$F_2 = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}$$

① $$\oint_{\Gamma} wF_1 n_x ds - \int_{\Omega} \frac{\partial w}{\partial x} F_1 dxdy$$

② $$\oint_{\Gamma} wF_2 n_y ds - \int_{\Omega} \frac{\partial w}{\partial y} F_2 dxdy$$

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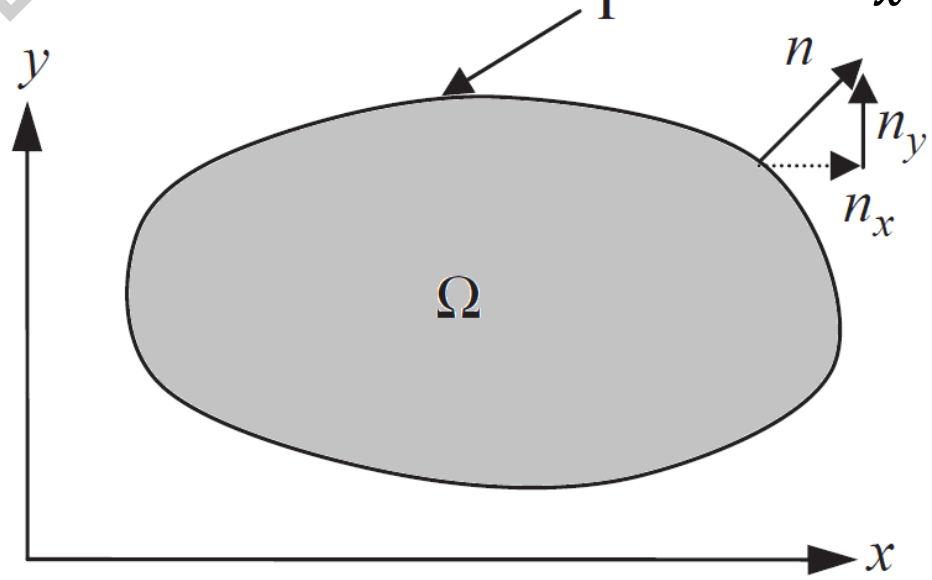
$$\hat{n} = \hat{n}_x\hat{\imath} + \hat{n}_x j$$



Weak Form

$$\int_{\Omega} w\left[\frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) - a_{00}u + f\right] dxdy = 0,$$

(1) (2) (3) (4)

$$F_1 = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}$$

$$F_2 = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}$$

① $\oint_{\Gamma} wF_1 n_x ds - \int_{\Omega} \frac{\partial w}{\partial x} F_1 dxdy$

② $\oint_{\Gamma} wF_2 n_y ds - \int_{\Omega} \frac{\partial w}{\partial y} F_2 dxdy$

③ $-\int_{\Omega} a_{00} wu dxdy$

④ $\int_{\Omega} wf dxdy$

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Weak Form

$$\int_\Omega w\left[\frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) - a_{00}u + f\right]dxdy = 0,$$

(1) (2) (3) (4)

$$F_1 = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}$$

$$F_2 = a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}$$

$$\int_\Omega \left[-\frac{\partial w}{\partial x}F_1 - \frac{\partial w}{\partial y}F_2 - a_{00}wu\right]dxdy + \oint_\Gamma wF_1 n_x ds + \oint_\Gamma wF_2 n_y ds + \int_\Omega wf\,dxdy = 0$$

(1a) (2a) (3) (1b) (2b) (4)

$$\Rightarrow -\int_\Omega \left[\frac{\partial w}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial w}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}wu\right]dxdy + \int_\Omega wf\,dxdy$$

$$+\oint_\Gamma w\left[n_x\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + n_y\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)\right]ds = 0$$

Natural boundary condition $\Longrightarrow q_n$

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Weak Form

$$-\int_{\Omega}\left[\frac{\partial w}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial w}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+a_{00}wu\right]dxdy$$

$$-B(w,u)$$

$$\int_{\Omega} wf\,dxdy+\oint_{\Gamma} wq_n\,ds=0$$

$$l(w)$$

$$B(w,u)=l(w)$$

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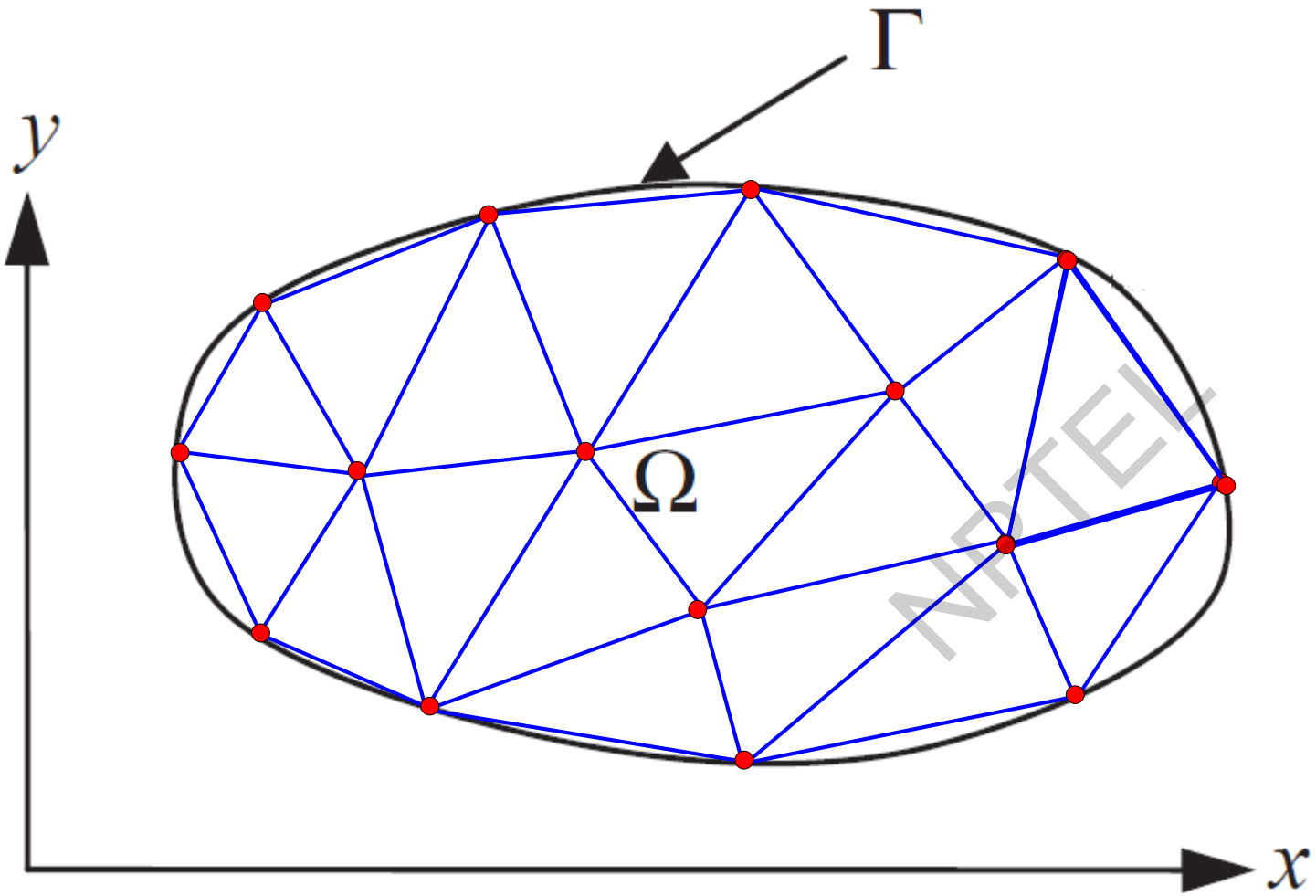
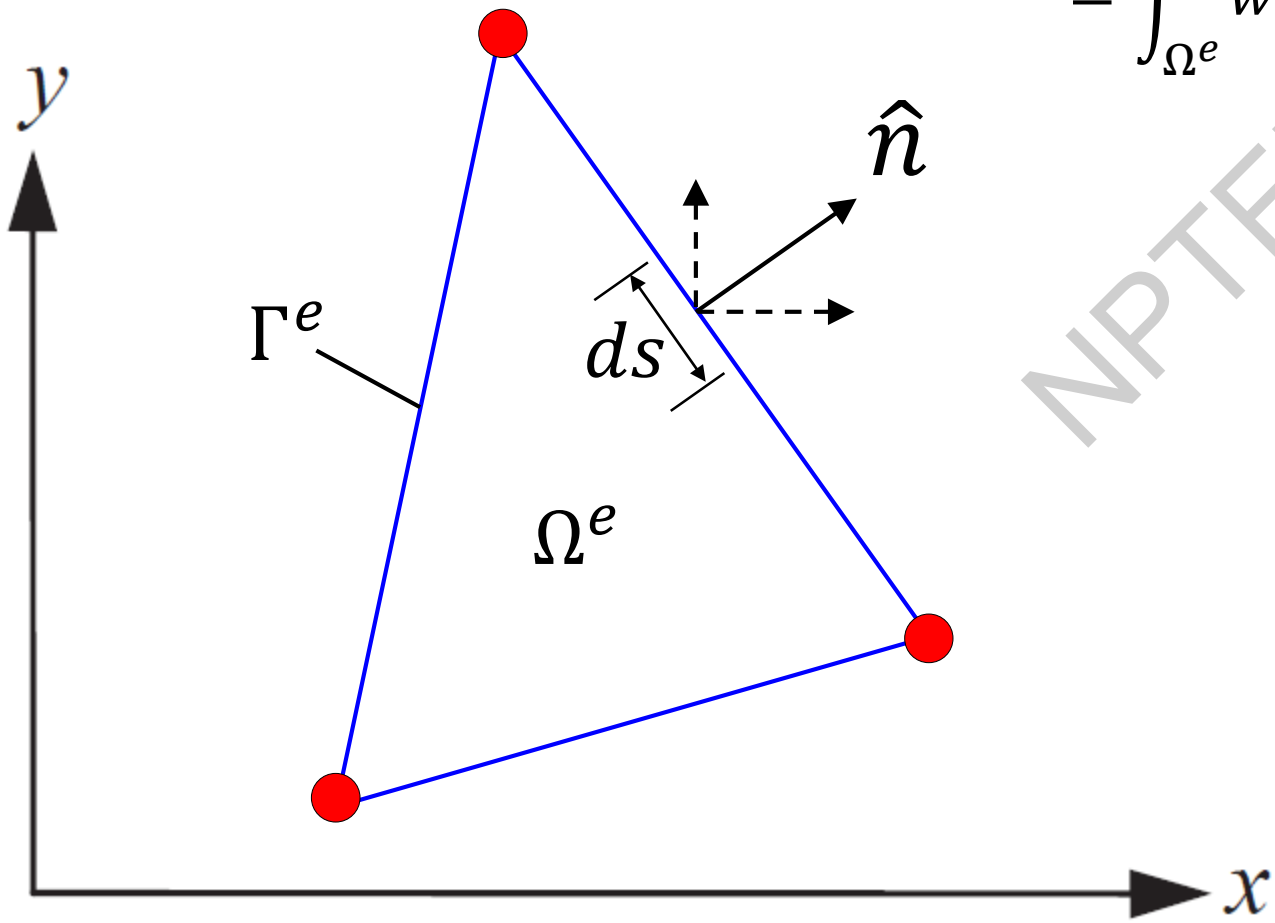
Finite Element Model



$$B^e(w,u) = l^e(w)$$

$$\int_{\Omega^e} \left[\frac{\partial w}{\partial x} \left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial y} \left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) + a_{00} w u \right] dx dy$$

$$= \int_{\Omega^e} w f \, dx dy + \oint_{\Gamma^e} w q_n \, ds$$



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$$u^e(x,y) = \sum_{i=1}^{n} \phi_i^e(x,y) u_i^e$$

Finite Element Model

$$\int_{\Omega^e}\left[\frac{\partial w}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial w}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+a_{00}wu\right]dxdy=\int_{\Omega^e}wf\,dxdy+\oint_{\Gamma^e}wq_n\,ds$$

$$\sum_{j=1}^{n}K_{ij}^{e}u_j^{e}$$

$$\sum_{j=1}^{n}\left\{\int_{\Omega^e}\left[\frac{\partial\phi_i^e}{\partial x}\left(a_{11}\frac{\partial\phi_j^e}{\partial x}+a_{12}\frac{\partial\phi_j^e}{\partial y}\right)+\frac{\partial\phi_i^e}{\partial y}\left(a_{21}\frac{\partial\phi_j^e}{\partial x}+a_{22}\frac{\partial\phi_j^e}{\partial y}\right)+a_{00}\phi_i^e\phi_j^e\right]dxdy\right\}u_j^e$$

$$=\underbrace{\int_{\Omega^e}f^e\phi_i^e\,dxdy}_{f_i^e}+\underbrace{\oint_{\Gamma^e}\phi_i^e q_n^e\,ds}_{Q_i^e}$$

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Finite Element Model

Specific case ($a_{11} = a_{22} = k$, $a_{00} = a_{12} = a_{21} = 0$): *Poisson equation*

$$\sum_{j=1}^{n} \left\{ \int_{\Omega^e} \left[\frac{\partial \phi_i^e}{\partial x} \left(k \frac{\partial \phi_j^e}{\partial x} \right) + \frac{\partial \phi_i^e}{\partial y} \left(k \frac{\partial \phi_j^e}{\partial y} \right) \right] dx dy \right\} u_j^e = \int_{\Omega^e} f \phi_i^e \, dx dy + \oint_{\Gamma^e} \phi_i^e q_n \, ds$$

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Finite Element Model

$$\sum_{j=1}^{n} K_{ij}^{e} u_j^{e} = f_i^{e} + Q_i^{e} \quad \Longrightarrow \quad [K^{e}]\{u^{e}\} = \{f^{e}\} + \{Q^{e}\}$$

$$\sum_{j=1}^{n} K_{ij}^{e} u_j^{e}$$

$$\sum_{j=1}^{n} \left\{ \int_{\Omega^e} \left[\frac{\partial \phi_i^e}{\partial x} \left(a_{11} \frac{\partial \phi_j^e}{\partial x} + a_{12} \frac{\partial \phi_j^e}{\partial y} \right) + \frac{\partial \phi_i^e}{\partial y} \left(a_{21} \frac{\partial \phi_j^e}{\partial x} + a_{22} \frac{\partial \phi_j^e}{\partial y} \right) + a_{00} \phi_i^e \phi_j^e \right] dx dy \right\} u_j^e$$

$$= \underbrace{\int_{\Omega^e} f^e \phi_i^e \, dx dy}_{f_i^e} + \underbrace{\oint_{\Gamma^e} \phi_i^e q_n^e \, ds}_{Q_i^e}$$

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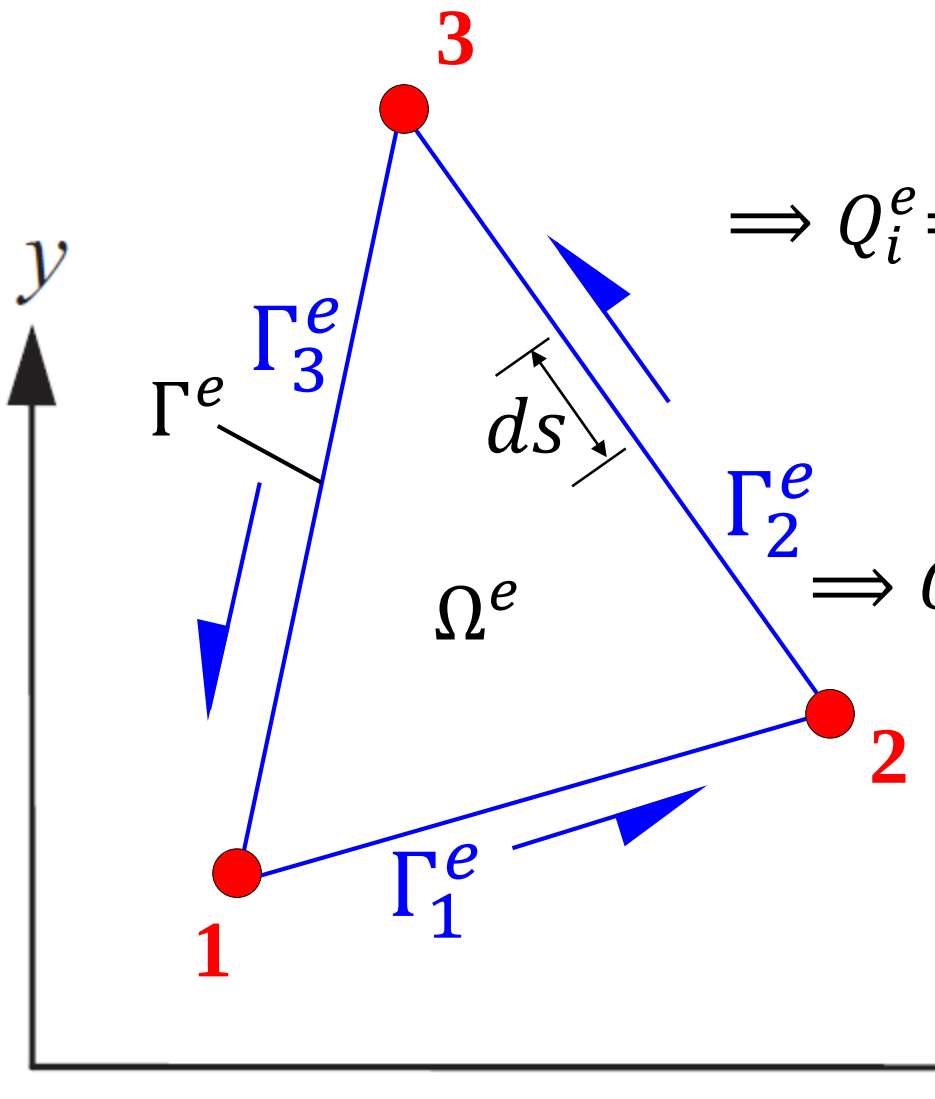
Evaluation of boundary integral

$$Q_i^e = \oint_{\Gamma^e} \phi_i^e q_n^e \, ds$$



$$\Longrightarrow Q_i^e = \oint_{\Gamma^1} \phi_i^e q_n^e \, ds + \oint_{\Gamma^2} \phi_i^e q_n^e \, ds + \oint_{\Gamma^3} \phi_i^e q_n^e \, ds$$

$$\Longrightarrow Q_i^e = \oint_{1-2} \phi_i^e q_n^e \, ds + \oint_{2-3} \phi_i^e q_n^e \, ds + \oint_{3-1} \phi_i^e q_n^e \, ds$$

$$\Longrightarrow Q_i^e = Q_{i1}^e + Q_{i2}^e + Q_{i3}^e$$

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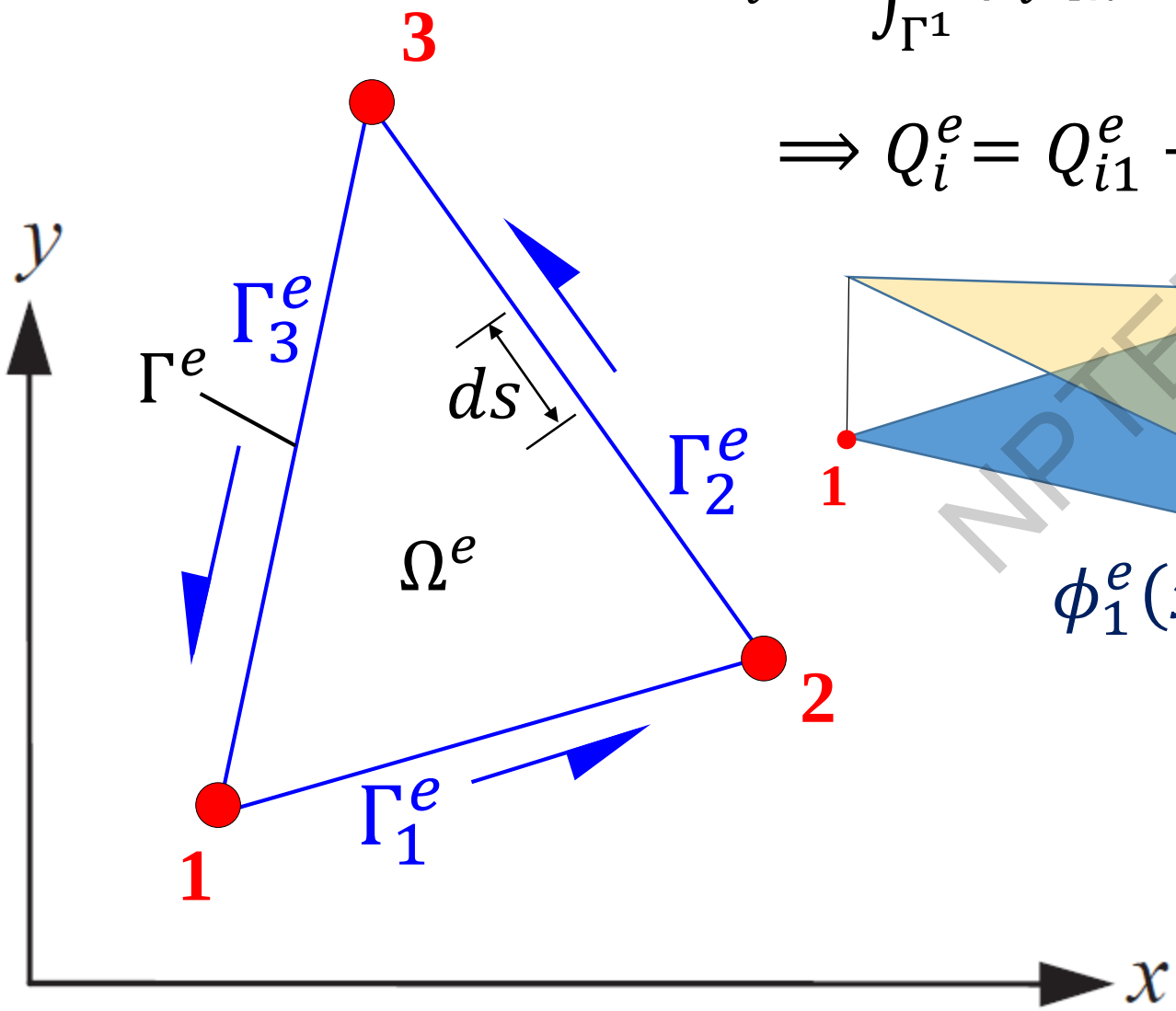
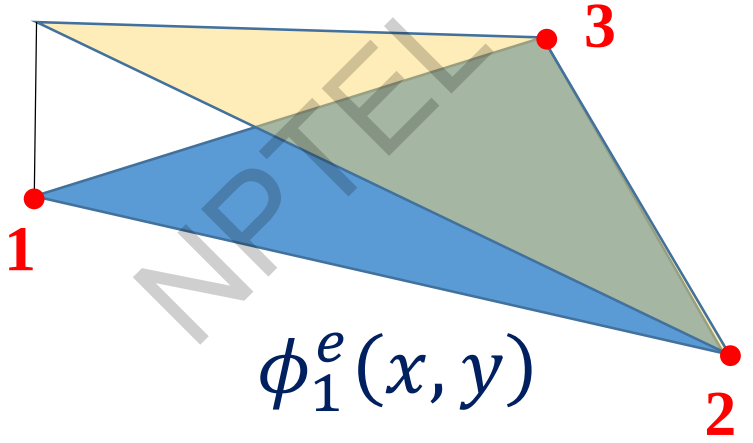
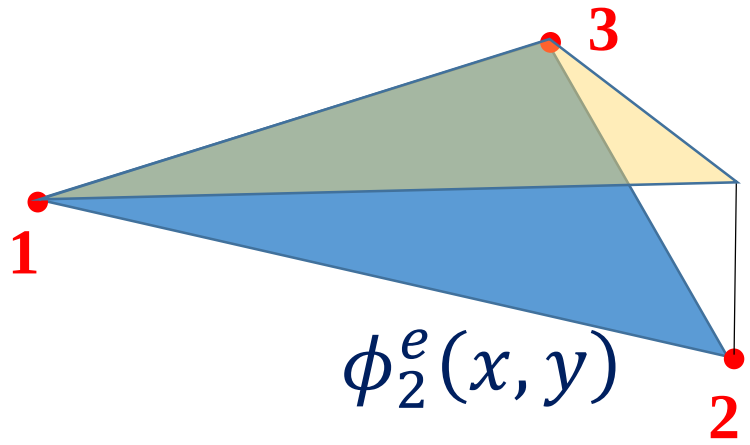
Evaluation of boundary integral

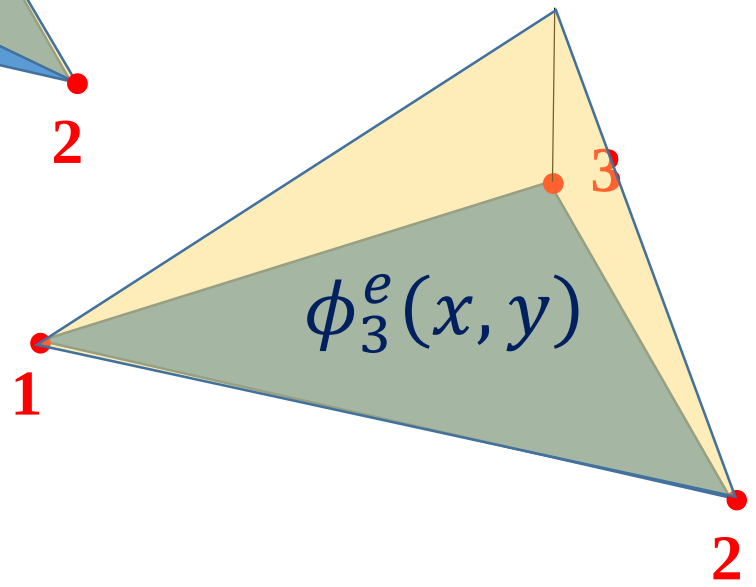
$$Q_i^e = \oint_{\Gamma^1} \phi_i^e q_n^e \, ds + \oint_{\Gamma^2} \phi_i^e q_n^e \, ds + \oint_{\Gamma^3} \phi_i^e q_n^e \, ds$$

$$\Rightarrow Q_i^e = Q_{i1}^e + Q_{i2}^e + Q_{i3}^e$$









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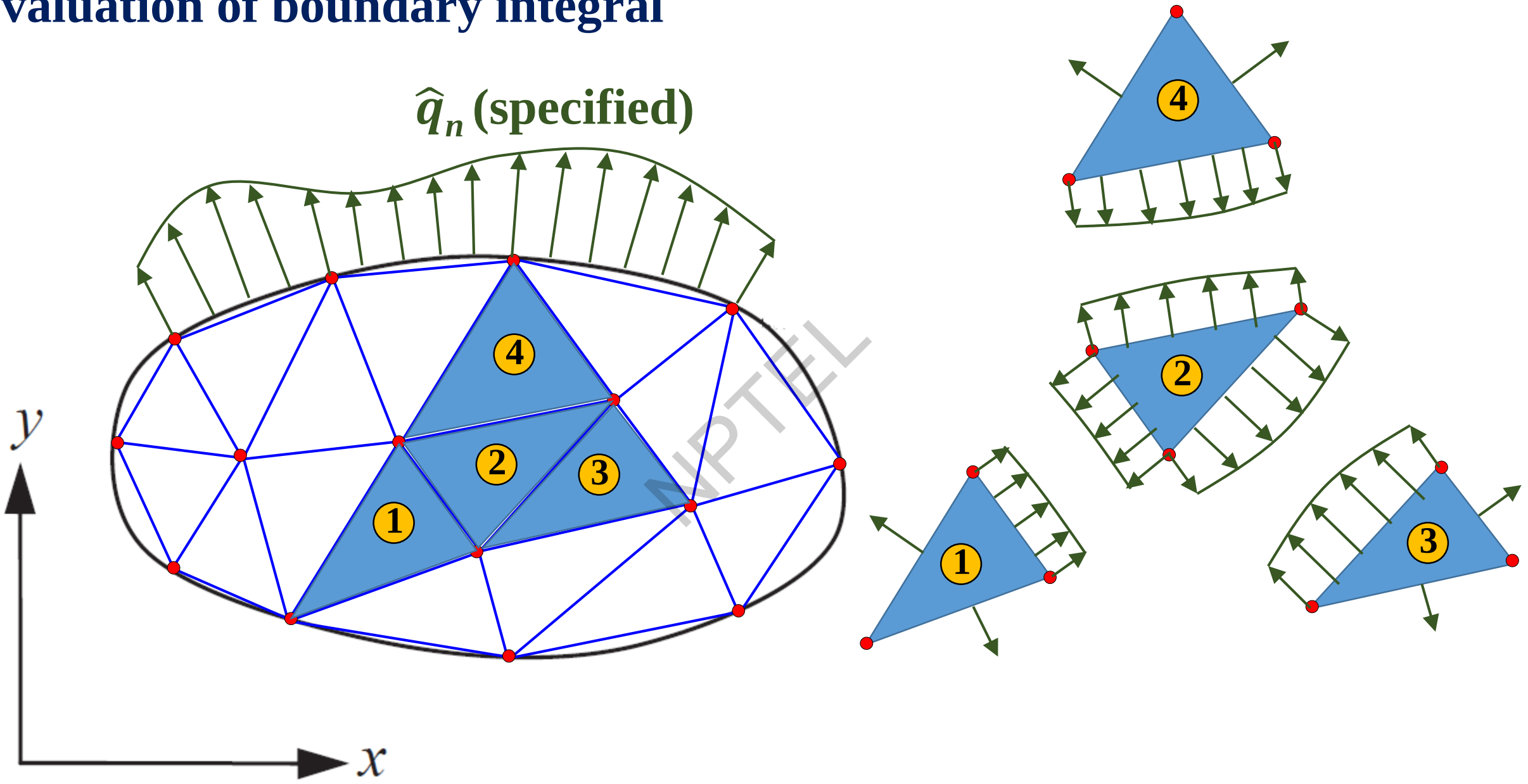
Evaluation of boundary integral



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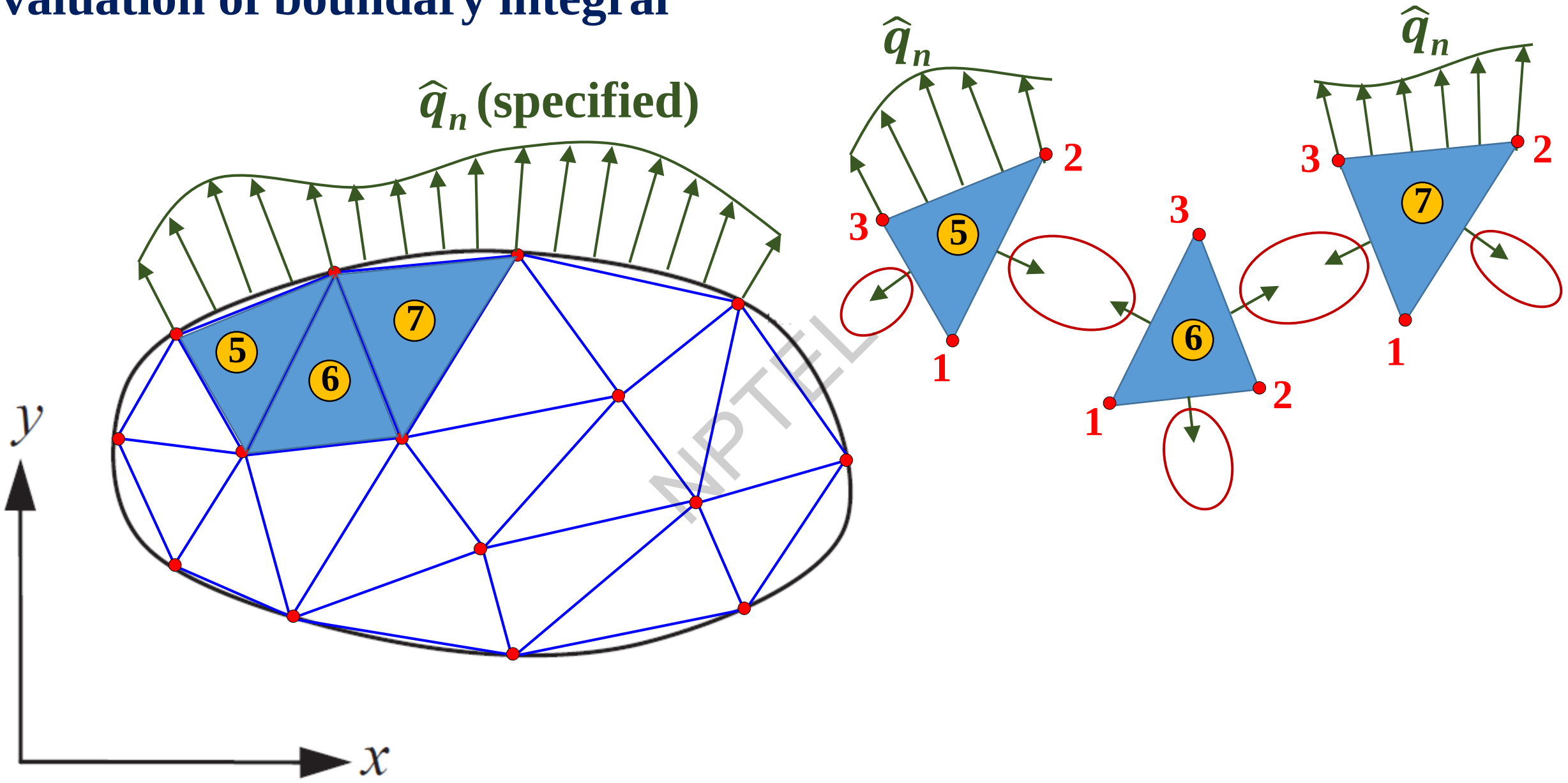
Evaluation of boundary integral



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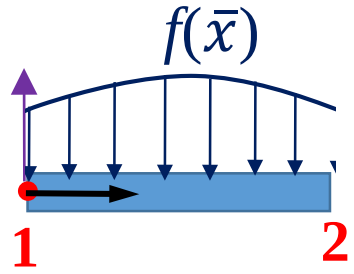
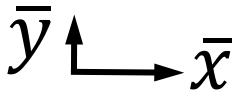
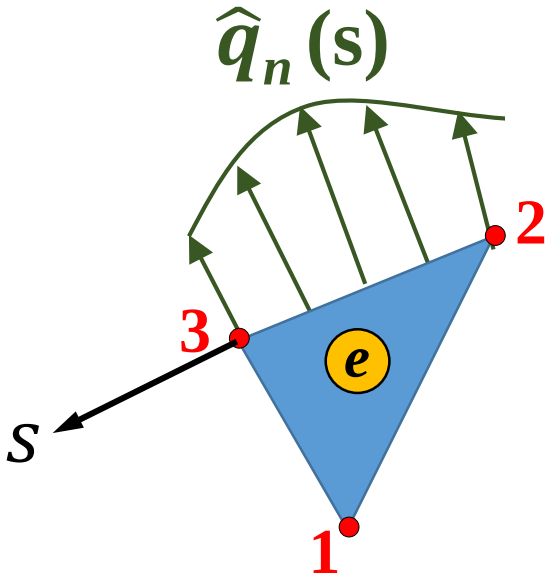
Recall: 1D Element

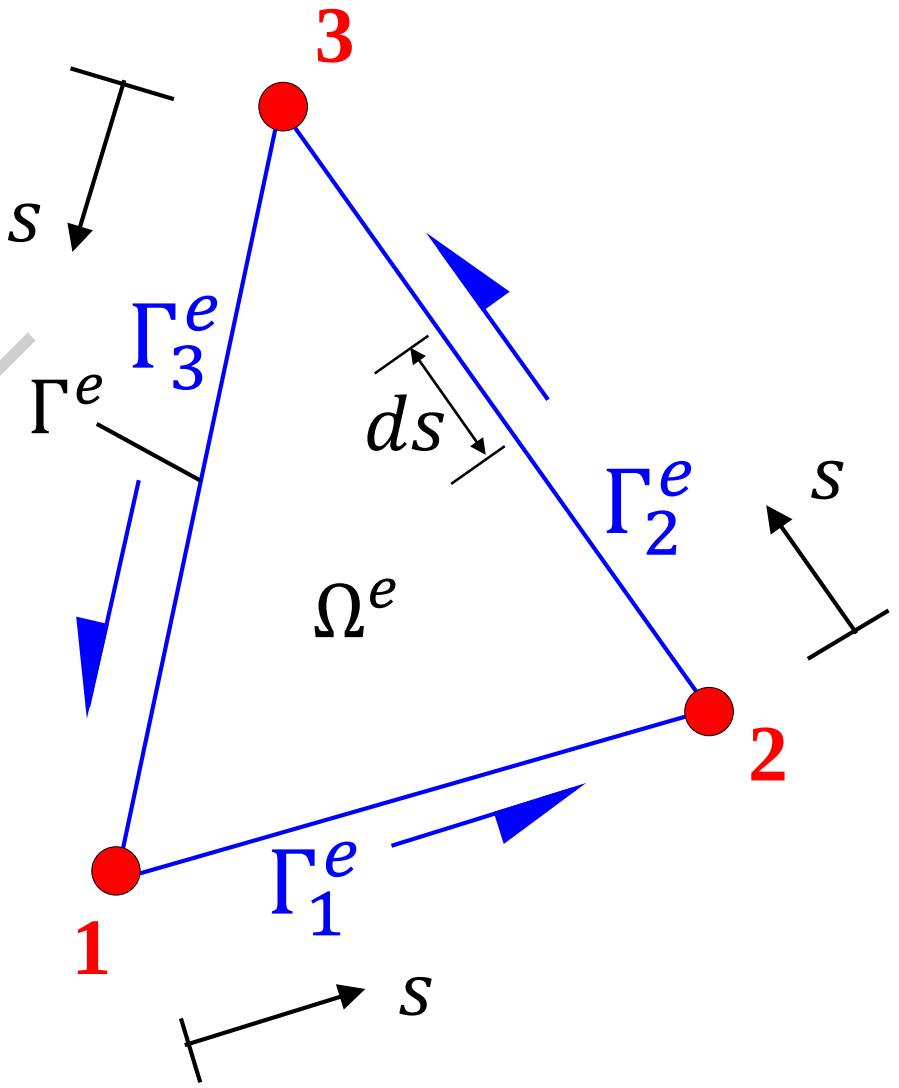


$$f_i^e = \int_{x_1}^{x_2} f(x)\phi_i^e \, dx$$



Element/ Local coordinate





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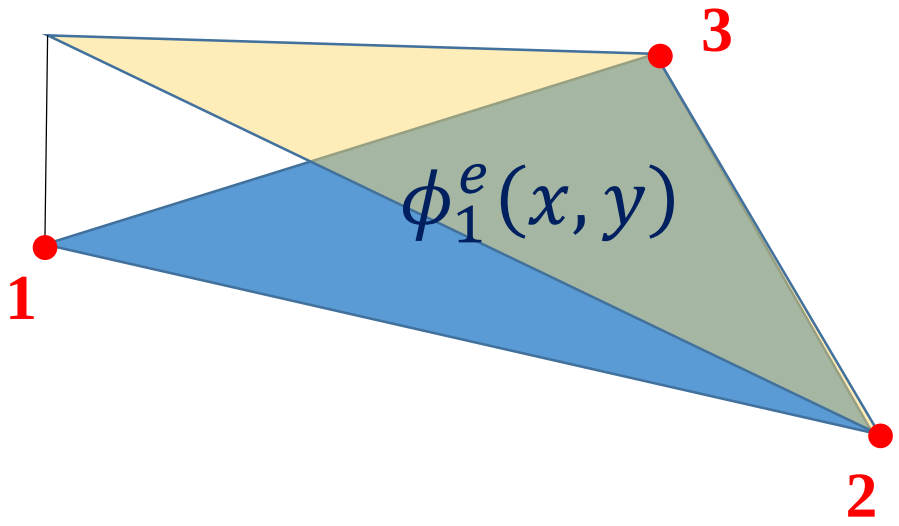
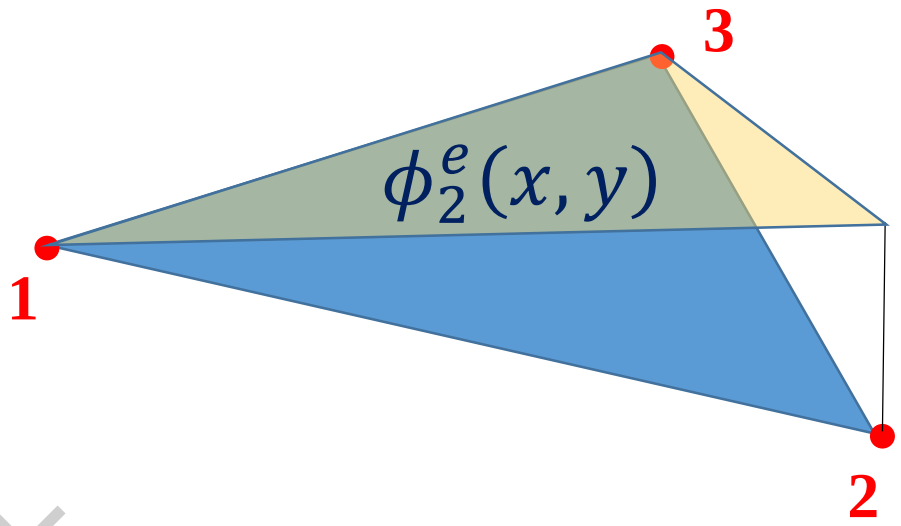
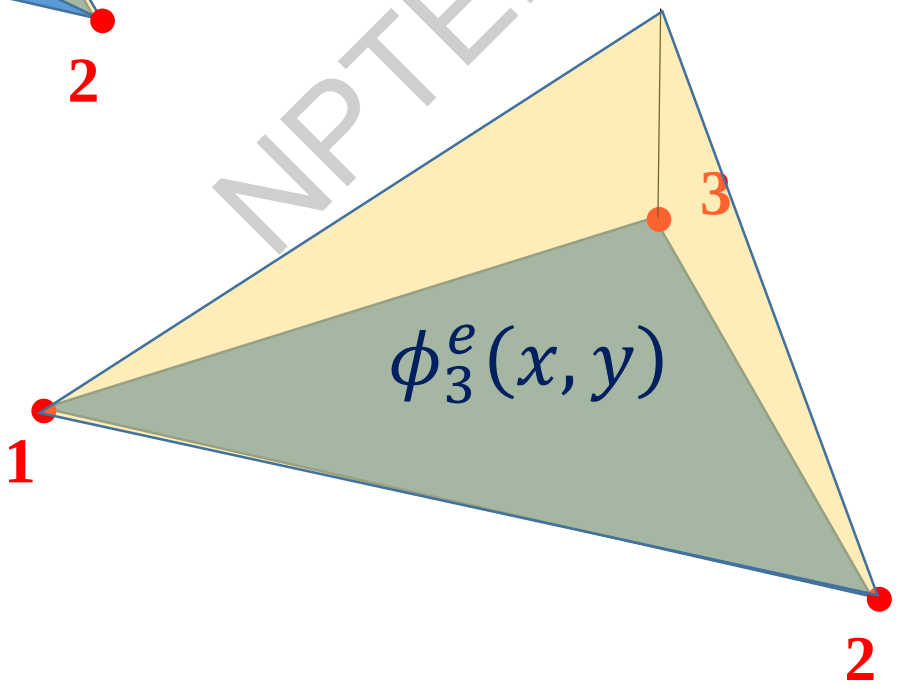
Evaluation of boundary integral







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THANK YOU !

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Finite Element Method

Week 8: Finite element formulation for 2D scalar field problems

Lecture 38: Example

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Recall

Strong Form

$$\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)-a_{00}u+f=0,$$

Weak Form

$$\int_{\Omega}\left[\frac{\partial w}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial w}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+a_{00}wu\right]dxdy$$

$$=\int_{\Omega}wf\,dxdy+\oint_{\Gamma}wq_n\,ds$$

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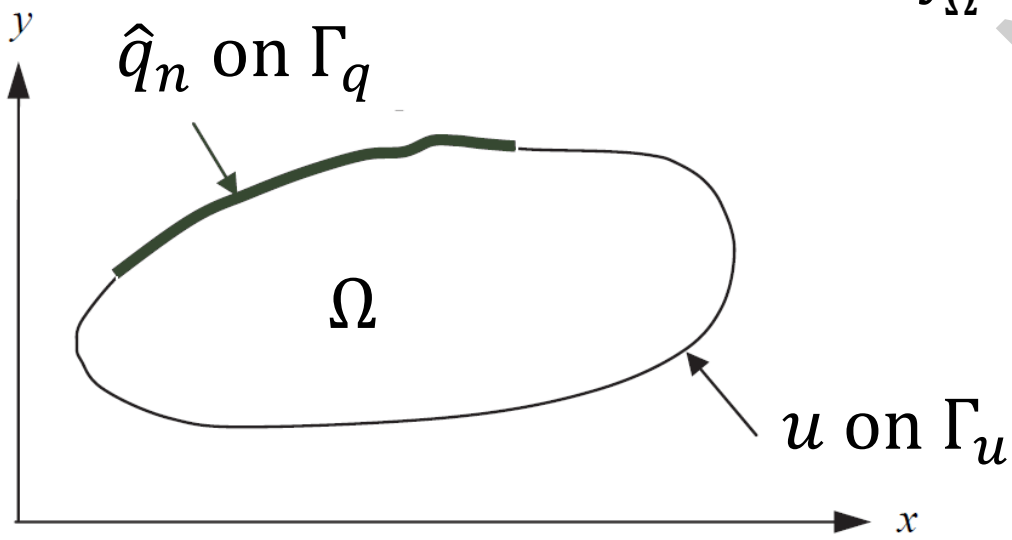

$$\Gamma_u \cup \Gamma_q = \Gamma$$

$$\Gamma_u \cap \Gamma_q = 0$$

$$\vec{n} = n_x\hat{\imath} + n_y\hat{\jmath}$$

$$\hat{q}_n = \hat{q}_x n_x + \hat{q}_y n_y$$

Recall

Strong Form

$$\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)-a_{00}u+f=0,$$

Weak Form and Finite Element Model

$$\sum_{j=1}^{n}\left\{\int_{\Omega^e}\left[\frac{\partial\phi_i^e}{\partial x}\left(a_{11}\frac{\partial\phi_j^e}{\partial x}+a_{12}\frac{\partial\phi_j^e}{\partial y}\right)+\frac{\partial\phi_i^e}{\partial y}\left(a_{21}\frac{\partial\phi_j^e}{\partial x}+a_{22}\frac{\partial\phi_j^e}{\partial y}\right)+a_{00}\phi_i^e\phi_j^e\right]dxdy\right\}u_j^e$$

$$\sum_{j=1}^{n}K_{ij}^e u_j^e = \int_{\Omega^e} f^e\phi_i^e\,dxdy + \oint_{\Gamma^e}\phi_i^e q_n^e\,ds$$

f_i^e $\qquad$ Q_i^e

$$\sum_{j=1}^{n}K_{ij}^e u_j^e = f_i^e + Q_i^e$$

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

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Recall

Specific case ($a_{11} = a_{22} = k$, $a_{00} = a_{12} = a_{21} = 0$): *Poisson equation*

$$\sum_{j=1}^{n}\left\{\int_{\Omega^e}\left[\frac{\partial\phi_i^e}{\partial x}\left(k\frac{\partial\phi_j^e}{\partial x}\right)+\frac{\partial\phi_i^e}{\partial y}\left(k\frac{\partial\phi_j^e}{\partial y}\right)\right]dxdy\right\}u_j^e = \int_{\Omega^e} f\phi_i^e\,dxdy + \oint_{\Gamma^e}\phi_i^e q_n\,ds$$

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e}\mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e dxdy \qquad \mathbf{C} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial\phi_1^e}{\partial x} & \frac{\partial\phi_2^e}{\partial x} & \cdots & \frac{\partial\phi_n^e}{\partial x} \\ \frac{\partial\phi_1^e}{\partial y} & \frac{\partial\phi_2^e}{\partial y} & \cdots & \frac{\partial\phi_n^e}{\partial y} \end{bmatrix}$$

n = number of nodes

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Recall

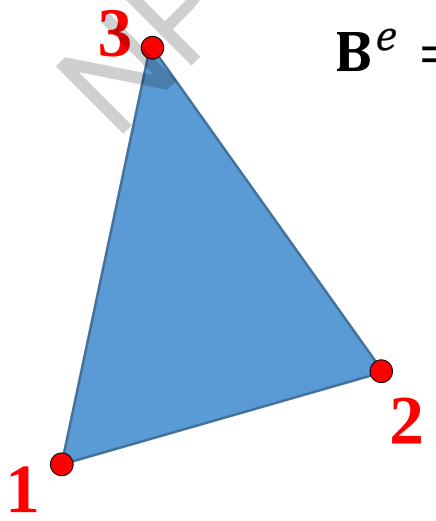
Specific case ($a_{11} = a_{22} = k$, $a_{00} = a_{12} = a_{21} = 0$): *Poisson equation*

$$\sum_{j=1}^{n}\left\{\int_{\Omega^e}\left[\frac{\partial\phi_i^e}{\partial x}\left(k\frac{\partial\phi_j^e}{\partial x}\right)+\frac{\partial\phi_i^e}{\partial y}\left(k\frac{\partial\phi_j^e}{\partial y}\right)\right]dxdy\right\}u_j^e = \int_{\Omega^e} f\phi_i^e\,dxdy + \oint_{\Gamma^e}\phi_i^e q_n\,ds$$

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e}\mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e dxdy \qquad \mathbf{C} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial\phi_1^e}{\partial x} & \frac{\partial\phi_2^e}{\partial x} & \cdots & \frac{\partial\phi_n^e}{\partial x} \\ \frac{\partial\phi_1^e}{\partial y} & \frac{\partial\phi_2^e}{\partial y} & \cdots & \frac{\partial\phi_n^e}{\partial y} \end{bmatrix}$$



$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial\phi_1^e}{\partial x} & \frac{\partial\phi_2^e}{\partial x} & \frac{\partial\phi_3^e}{\partial x} \\ \frac{\partial\phi_1^e}{\partial y} & \frac{\partial\phi_2^e}{\partial y} & \frac{\partial\phi_3^e}{\partial y} \end{bmatrix} \qquad \{u^e\} = \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix}$$

$$\{f^e\} = \begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \end{Bmatrix} \qquad \{Q^e\} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix}$$

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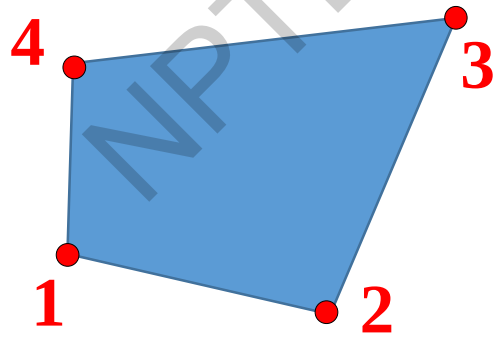
Recall

Specific case ($a_{11} = a_{22} = k$, $a_{00} = a_{12} = a_{21} = 0$): *Poisson equation*

$$\sum_{j=1}^{n}\left\{\int_{\Omega^e}\left[\frac{\partial\phi_i^e}{\partial x}\left(k\frac{\partial\phi_j^e}{\partial x}\right)+\frac{\partial\phi_i^e}{\partial y}\left(k\frac{\partial\phi_j^e}{\partial y}\right)\right]dxdy\right\}u_j^e=\int_{\Omega^e}f\phi_i^e\,dxdy+\oint_{\Gamma^e}\phi_i^e q_n\,ds$$

$$[K^e]\{u^e\}=\{f^e\}+\{Q^e\}$$

$$[K^e]=\int_{\Omega^e}\mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e\,dxdy \qquad \mathbf{C}=\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$$



NPTEL

$$\mathbf{B}^e=\begin{bmatrix}\frac{\partial\phi_1^e}{\partial x} & \frac{\partial\phi_2^e}{\partial x} & \frac{\partial\phi_3^e}{\partial x} & \frac{\partial\phi_4^e}{\partial x}\\ \frac{\partial\phi_1^e}{\partial y} & \frac{\partial\phi_2^e}{\partial y} & \frac{\partial\phi_3^e}{\partial y} & \frac{\partial\phi_4^e}{\partial y}\end{bmatrix}$$

$$\mathbf{B}^e=\begin{bmatrix}\frac{\partial\phi_1^e}{\partial x} & \frac{\partial\phi_2^e}{\partial x} & \cdots & \frac{\partial\phi_n^e}{\partial x}\\ \frac{\partial\phi_1^e}{\partial y} & \frac{\partial\phi_2^e}{\partial y} & \cdots & \frac{\partial\phi_n^e}{\partial y}\end{bmatrix}$$

$$\{u^e\}=\begin{Bmatrix}u_1^e\\ u_2^e\\ u_3^e\\ u_4^e\end{Bmatrix} \qquad \{f^e\}=\begin{Bmatrix}f_1^e\\ f_2^e\\ f_3^e\\ f_4^e\end{Bmatrix} \qquad \{Q^e\}=\begin{Bmatrix}Q_1^e\\ Q_2^e\\ Q_3^e\\ Q_4^e\end{Bmatrix}$$

$$\nabla \cdot (k\nabla u) = -f \text{ in } \Omega$$

Governing equations

$$u = \hat{u} \text{ on } \Gamma_u$$

Essential BC

$$k\,\partial u/\partial n + \beta(u - u_\infty) = q \text{ on} \Gamma_q$$

Natural BC

Field of applications	Primary variable *u*	Material constant *k*	Source variable *f*	Secondary variable *q*, $\partial u/\partial x$, $\partial u/\partial y$
Heat Transfer	Temperature T	Conductivity *k*	Heat source *g*	Heat flow due to conduction $k\partial T/\partial n$ Convection $h(T - T_\infty)$
Irrotational flow of an ideal fluid	Stream function Ψ Velocity potential Φ	Density (ρ)	Mass production σ	Velocities $\partial\Psi/\partial x = -v$ $\partial\Psi/\partial y = u$ $\partial\Phi/\partial x = u$ $\partial\Phi/\partial y = v$
Ground water flow	Piezometric head Φ	Permeability *K*	Recharge *f* (pumping, $-f$)	Seepage $q = k\partial\Phi/\partial n$ Velocities $k\partial\Phi/\partial x = -u$ $k\partial\Phi/\partial y = -v$
Torsion of cylindrical members with constant cross-section	Stress function Ψ	$k = 1$ G = shear modulus	$f = 2$ θ = angle of twist per unit length	$G\theta\partial\Psi/\partial x = -\sigma_{yz}$ $G\theta\partial\Psi/\partial y = \sigma_{xz}$
Electrostatics	Scalar potential Φ	Dielectric constant (ε)	Charge density ρ	Displacement flux D_n
Magnetostatics	Magnetic potential Φ	Permeability (μ)	Charge density ρ	Displacement flux B_n
Transverse deflection of elastic membrane	Transverse deflection *u*	Tension in member *T*	Transversely distributed load	Normal force *q*

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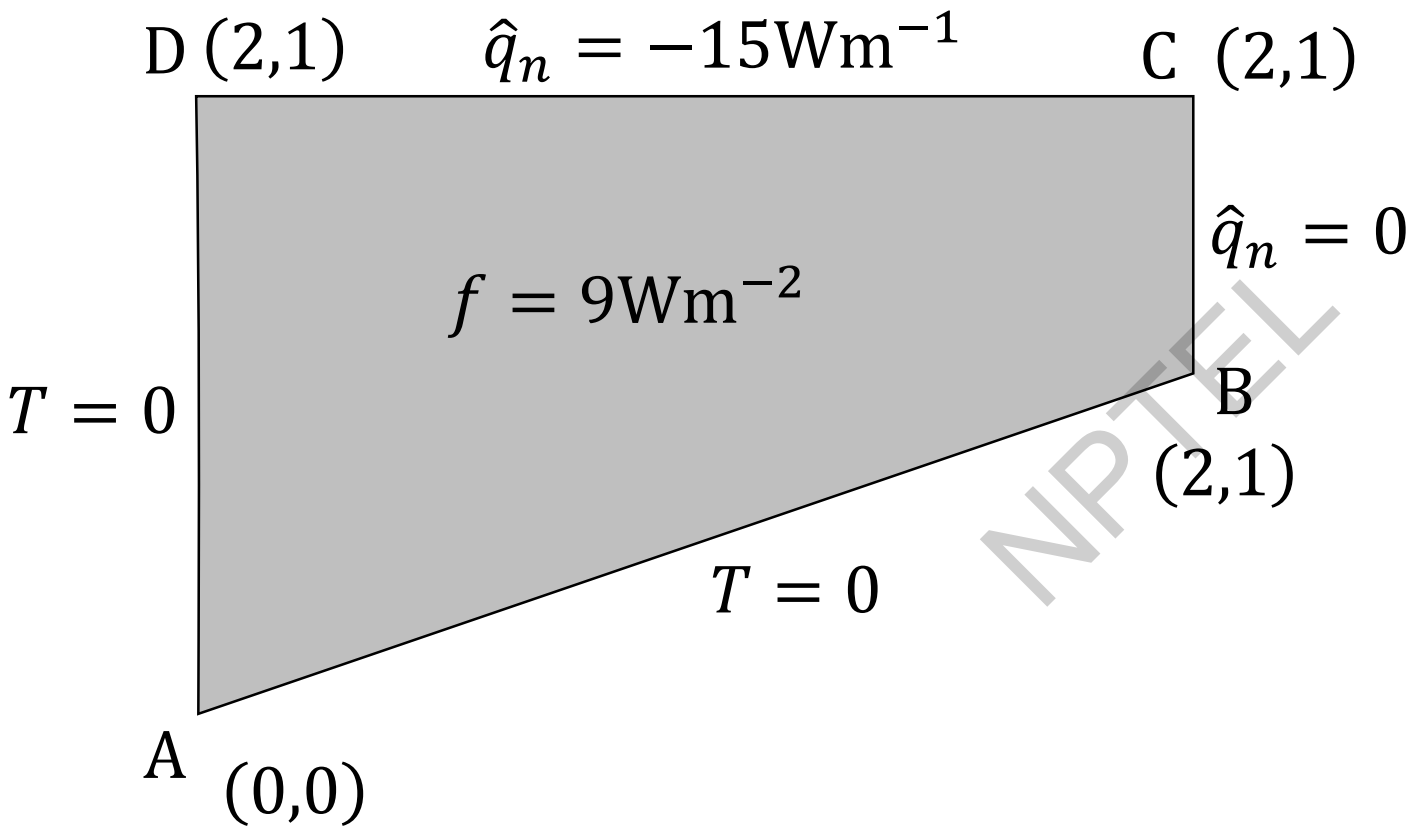
Example 1: Heat Conduction Problem $\nabla \cdot (k\nabla T) = -f$



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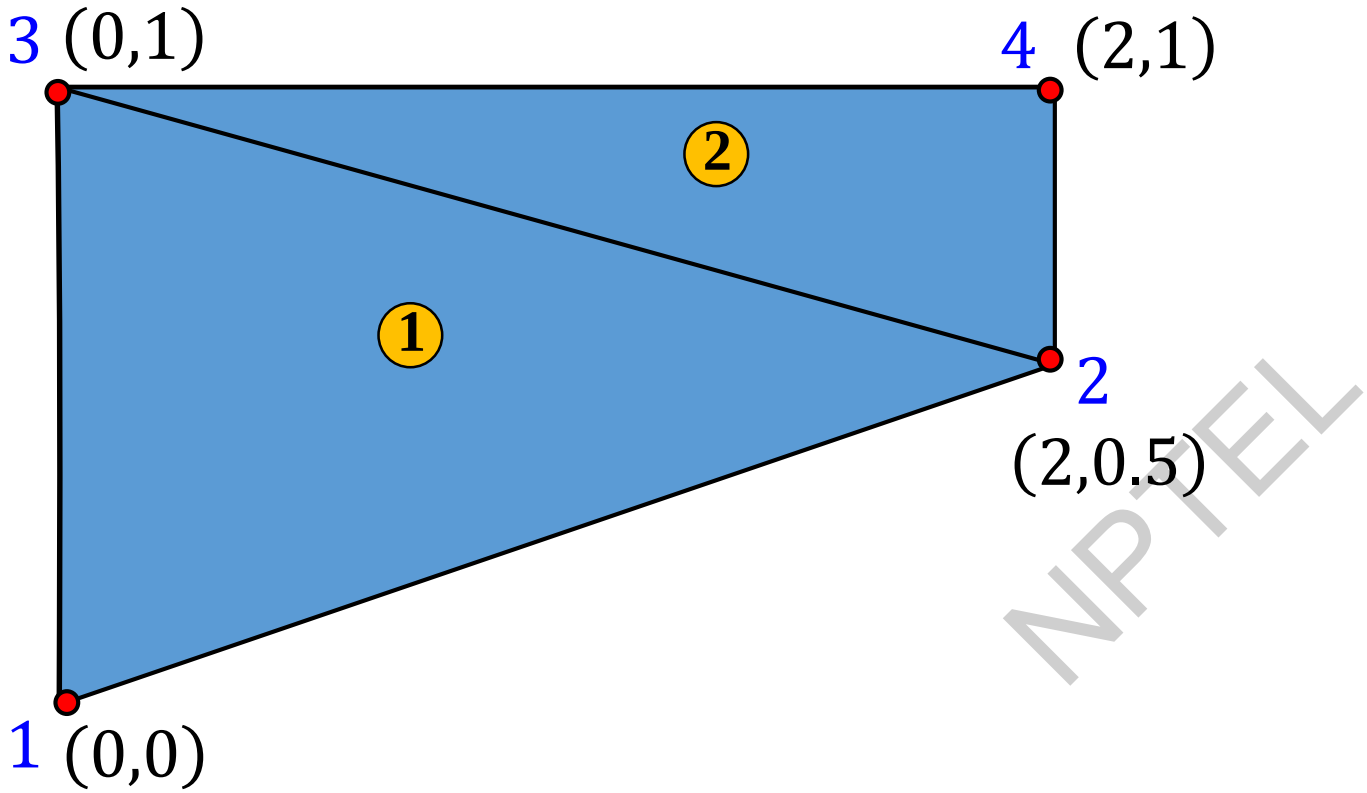
Step 1: Discretization



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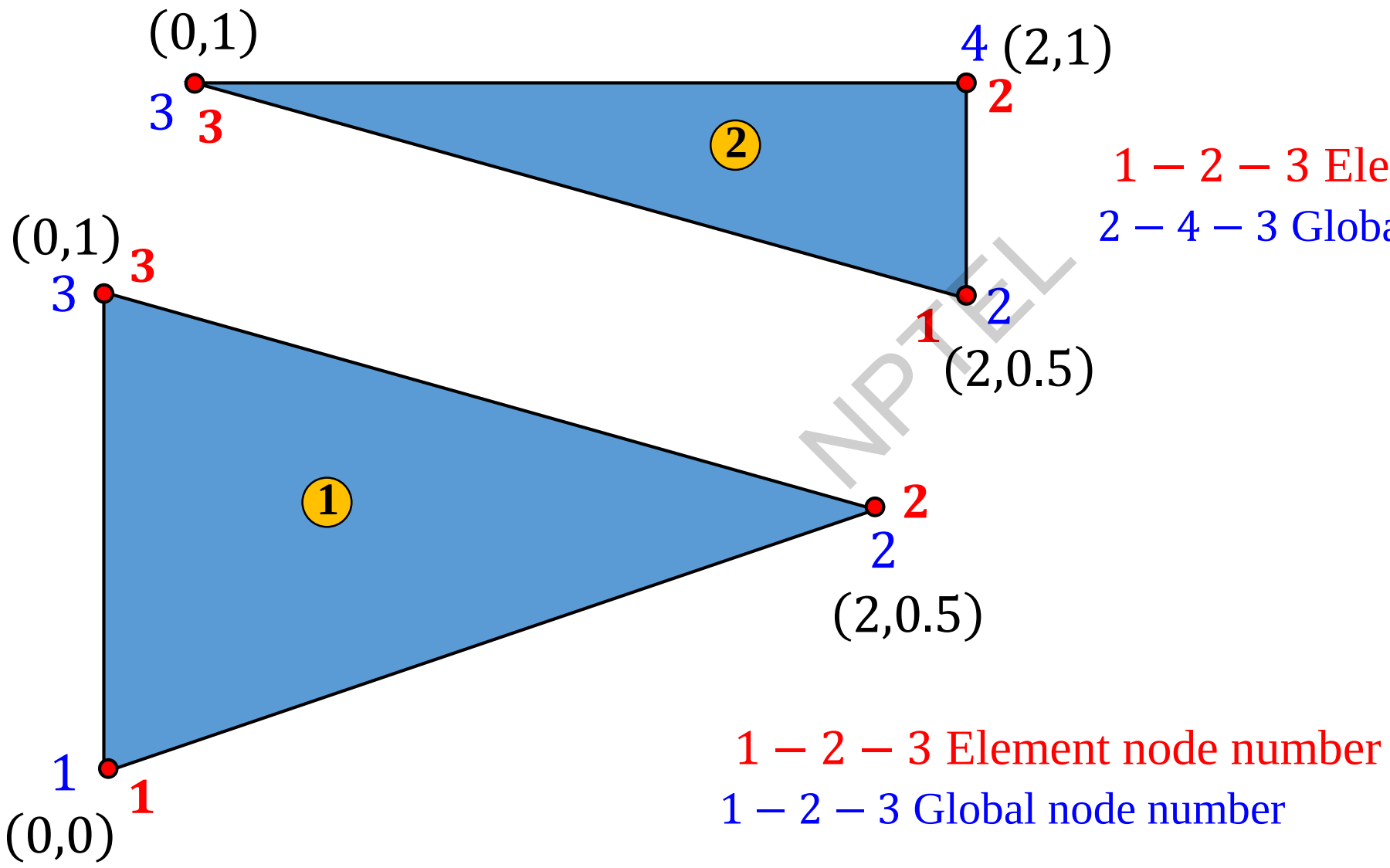
Step 2: Element Equation



1 – 2 – 3 Element node number

2 – 4 – 3 Global node number

1 – 2 – 3 Element node number

1 – 2 – 3 Global node number

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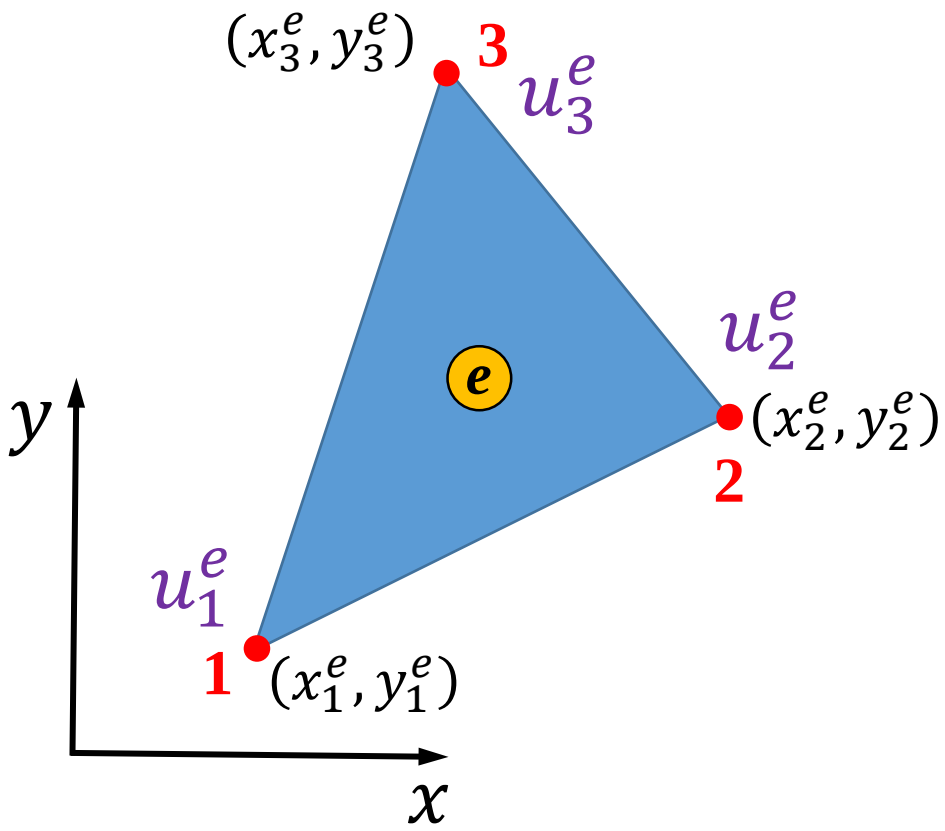
Recall: Shape Function For 3-Node Triangle



Polynomial approximation

$$u^e(x,y) = c_1 + c_2 x + c_3 y$$

$$u^e(x,y) = \{1 \quad x \quad y\} \begin{bmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{bmatrix}^{-1} \begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

$$2A^e = |\mathbf{J}|$$

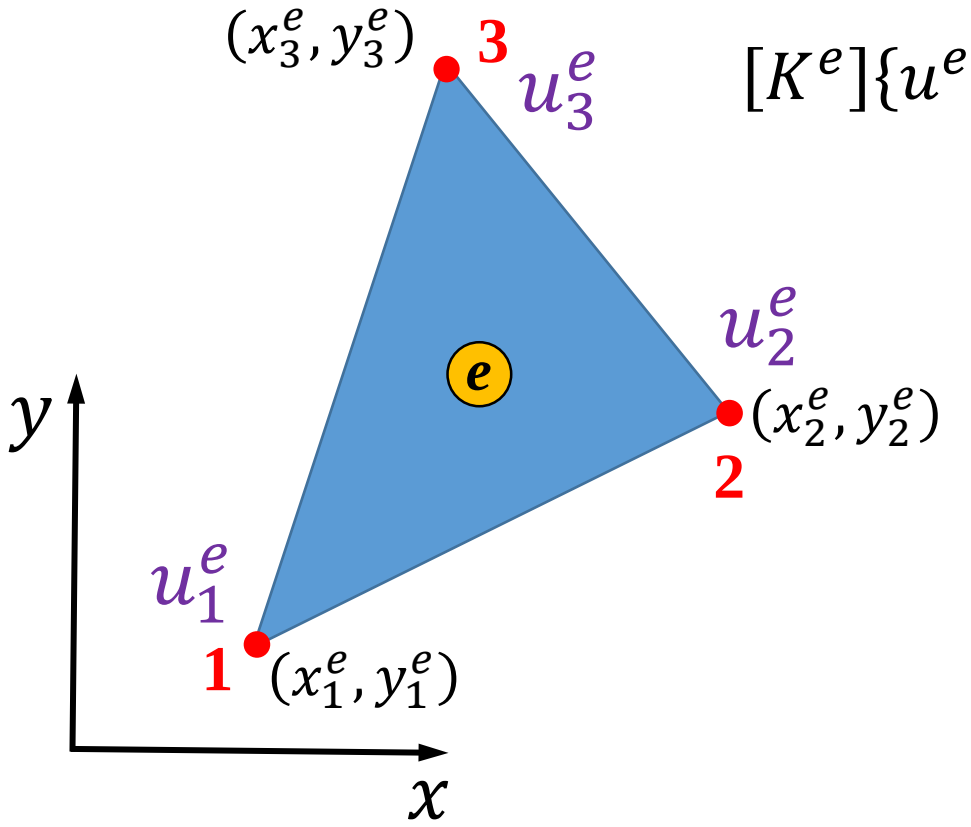
$$\phi_1^e = \frac{1}{2A^e}\{x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y\}$$

$$\phi_2^e = \frac{1}{2A^e}\{x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y\}$$

$$\phi_3^e = \frac{1}{2A^e}\{x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y\}$$

NPTEL

Example 1: Heat Conduction Problem



$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{e\mathrm{T}} \mathbf{C} \mathbf{B}^e \, dxdy$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \frac{\partial \phi_3^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \frac{\partial \phi_3^e}{\partial y} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

$$2A^e = |\mathbf{J}|$$

$$\phi_1^e = \frac{1}{2A^e}\{x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y\}$$

$$\phi_2^e = \frac{1}{2A^e}\{x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y\}$$

$$\phi_3^e = \frac{1}{2A^e}\{x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y\}$$

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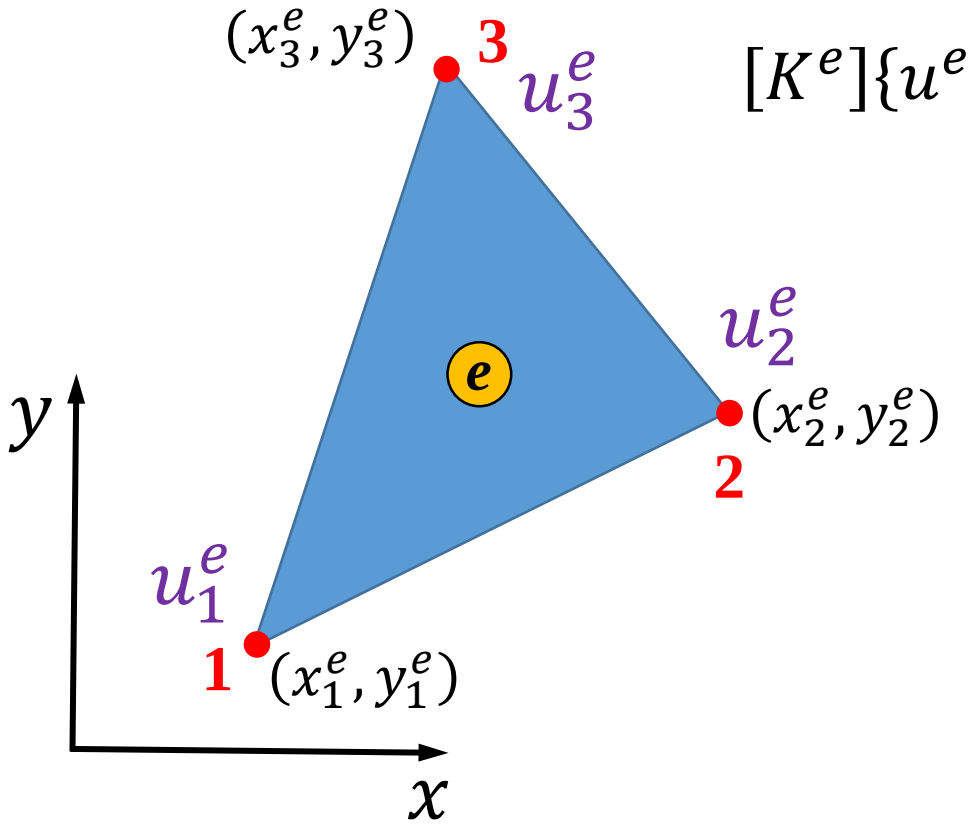
Example 1: Heat Conduction Problem



$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e \, dxdy$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \frac{\partial \phi_3^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \frac{\partial \phi_3^e}{\partial y} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{B}^e = \frac{1}{2A^e} \begin{bmatrix} (y_2^e - y_3^e) & (y_3^e - y_1^e) & (y_1^e - y_2^e) \\ (x_3^e - x_2^e) & (x_1^e - x_3^e) & (x_2^e - x_1^e) \end{bmatrix}$$

Constant

Recall: Constant strain triangle

$$\mathbf{J} = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

$$2A^e = |\mathbf{J}|$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e \, dxdy = \mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e \int_{\Omega^e} dxdy = A^e k \mathbf{B}^{e\mathrm{T}}\mathbf{B}^e$$

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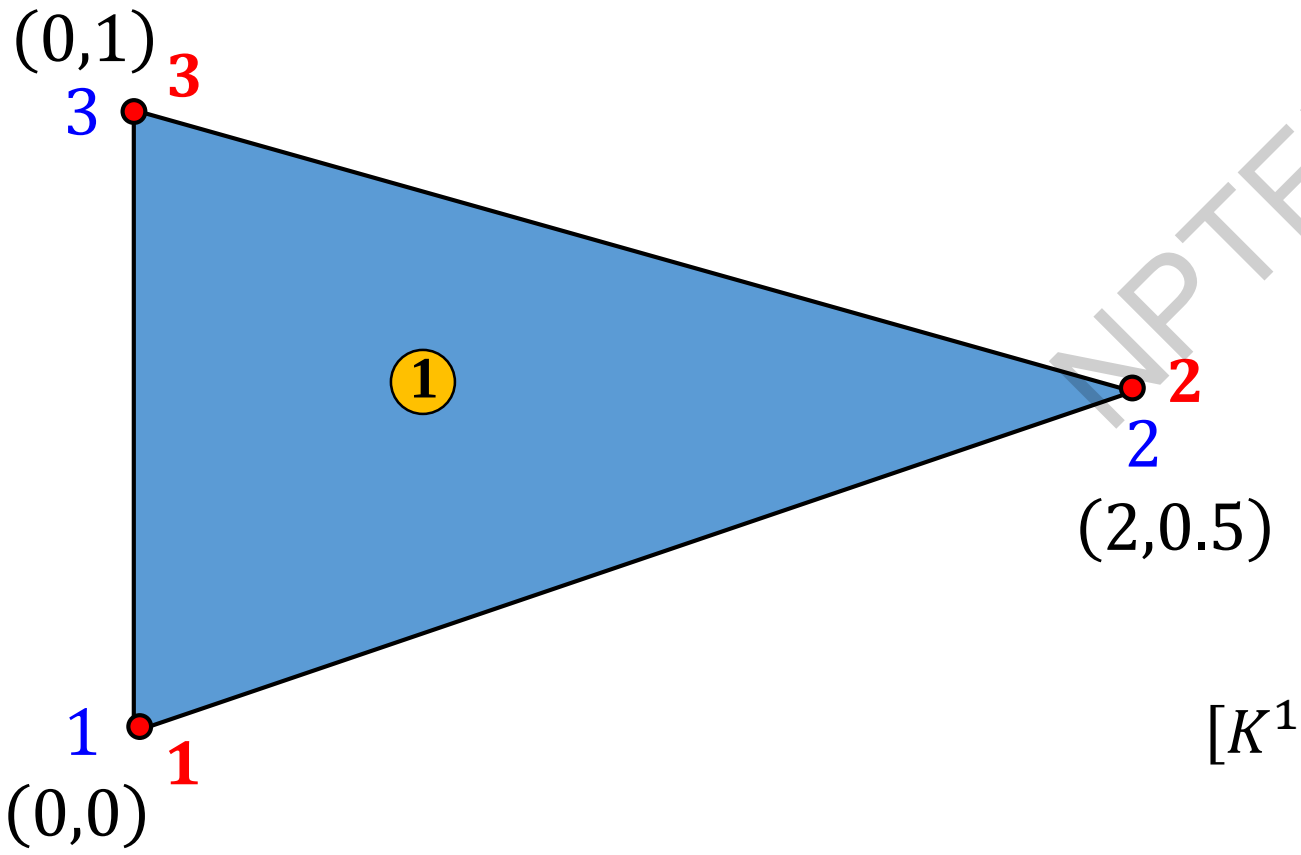
Step 2: Element Equation (Coefficient Matrix $[K^e]$)

$$\mathbf{B}^e = \frac{1}{2A^e}\begin{bmatrix}(y_2^e - y_3^e) & (y_3^e - y_1^e) & (y_1^e - y_2^e)\\ (x_3^e - x_2^e) & (x_1^e - x_3^e) & (x_2^e - x_1^e)\end{bmatrix}$$



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$$[B^1] = \begin{bmatrix}-0.25 & 0.5 & -0.25\\ -1 & 0 & 1\end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix}0 & -1\\ 2 & -0.5\end{bmatrix}$$

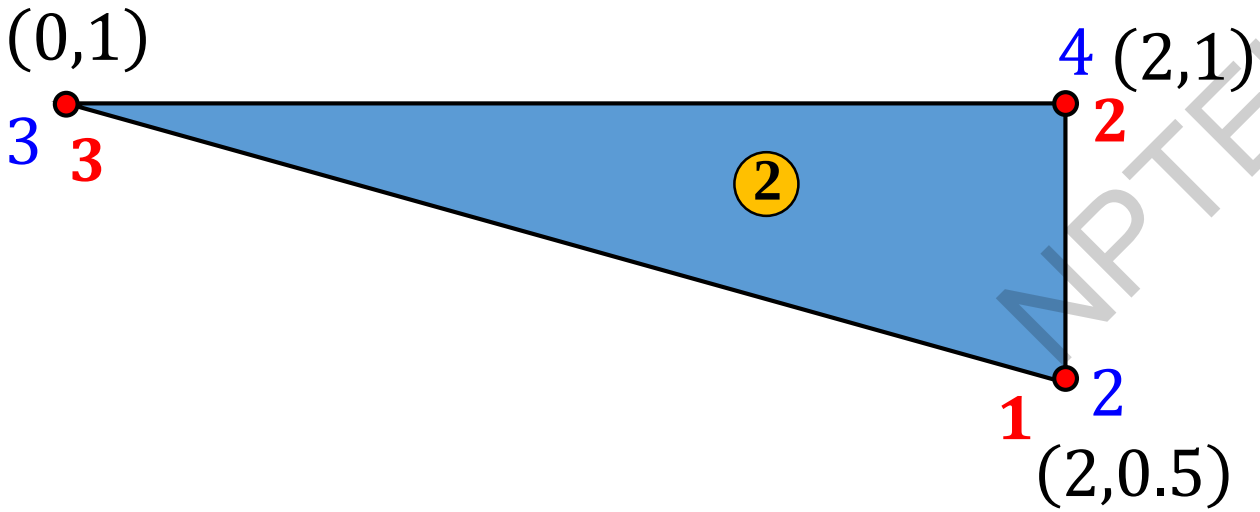
$$A^1 = \frac{1}{2}|\mathbf{J}| = 1$$

$$[K^1] = A^1 k \mathbf{B}^{1\mathrm{T}}\mathbf{B}^1$$

$$[K^1] = \begin{array}{c} \\ 1\\ 2\\ 3\end{array}\begin{array}{c}\begin{bmatrix}7.4375 & -0.8750 & -6.5625\\ -0.8750 & 1.7500 & -0.8750\\ -6.5625 & -0.8750 & 7.4375\end{bmatrix}\\ \begin{array}{ccc}1 & 2 & 3\end{array}\end{array}$$

Step 2: Element Equation (Coefficient Matrix $[K^e]$)

$$\mathbf{B}^e = \frac{1}{2A^e}\begin{bmatrix}(y_2^e - y_3^e) & (y_3^e - y_1^e) & (y_1^e - y_2^e)\\ (x_3^e - x_2^e) & (x_1^e - x_3^e) & (x_2^e - x_1^e)\end{bmatrix}$$



$$[B^1] = \begin{bmatrix}0 & 0.5 & -0.5\\ -2 & 2 & 0\end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix}2 & -0.5\\ 2 & 0\end{bmatrix}$$

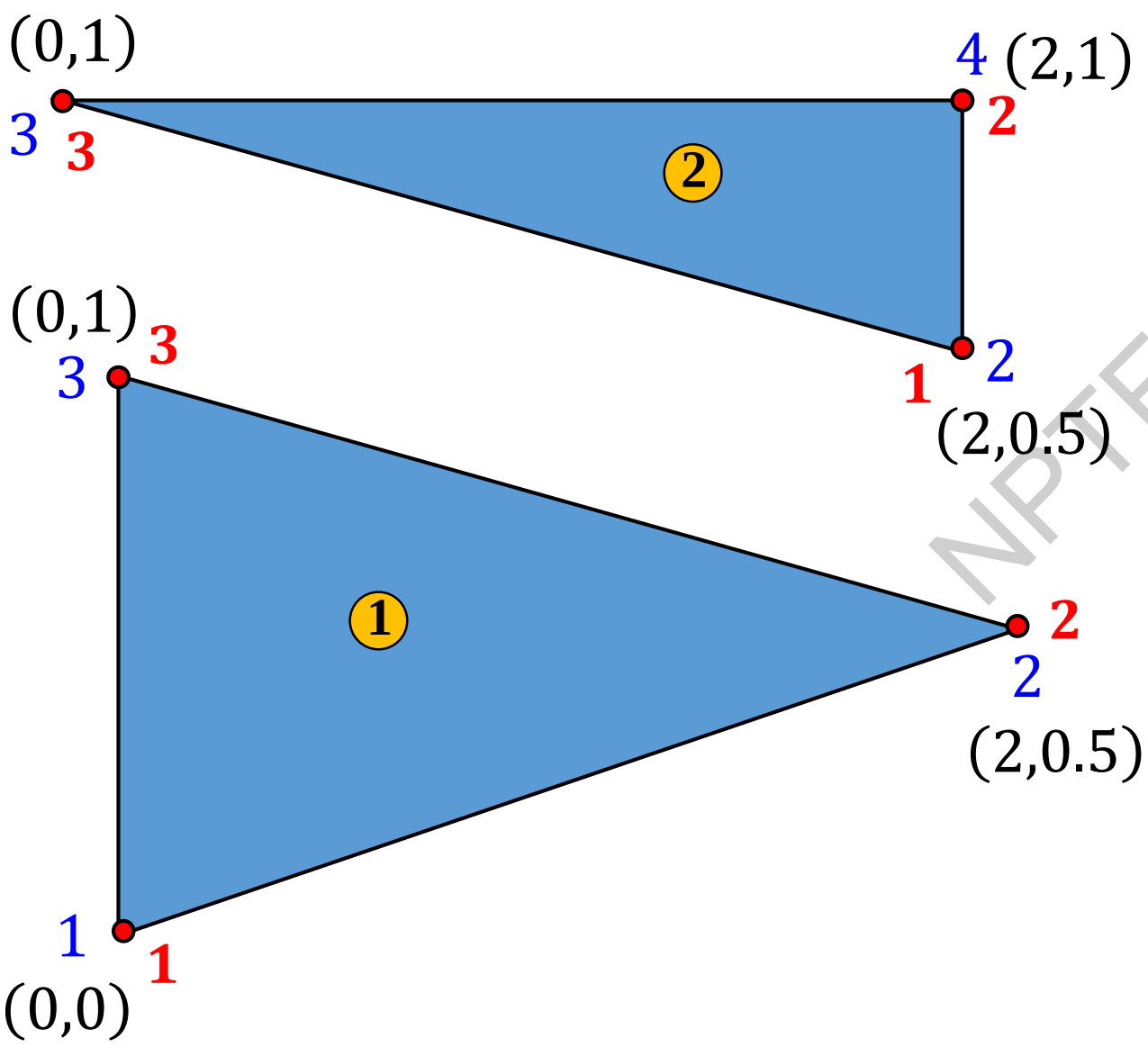
$$A^1 = \frac{1}{2}|\mathbf{J}| = 0.5$$

$$[K^1] = A^1 k \mathbf{B}^{1\mathrm{T}}\mathbf{B}^1$$

$$[K^2] = \begin{array}{c} \\ 2 \\ 4 \\ 3 \end{array}\begin{array}{c}\begin{array}{ccc} 2 & 4 & 3 \end{array}\\ \begin{bmatrix}14 & -14 & 0\\ -14 & 14.8750 & -0.8750\\ 0 & -0.8750 & 0.8750\end{bmatrix}\end{array}$$

NPTEL

Step 2: Element Equation (Source Term $\{f^e\}$)



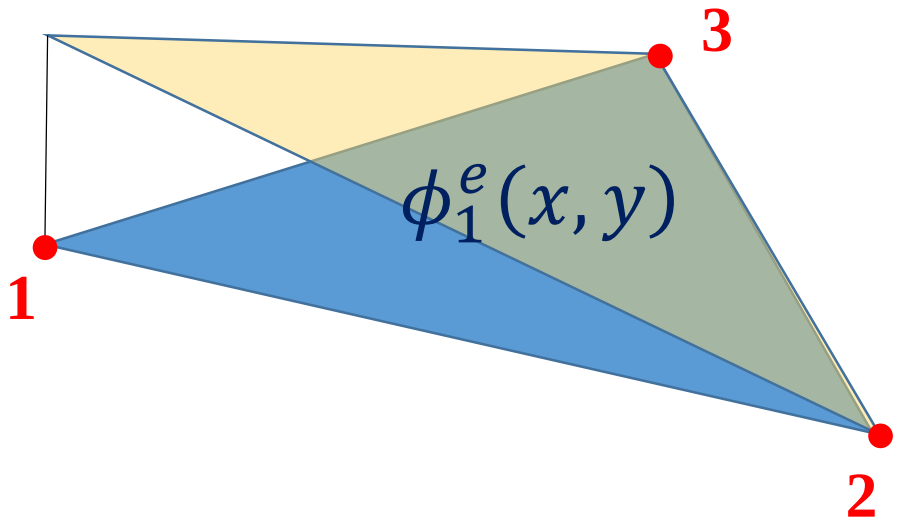
$$\{f^e\} = \int_{\Omega^e} f \phi_i^e \, dxdy$$

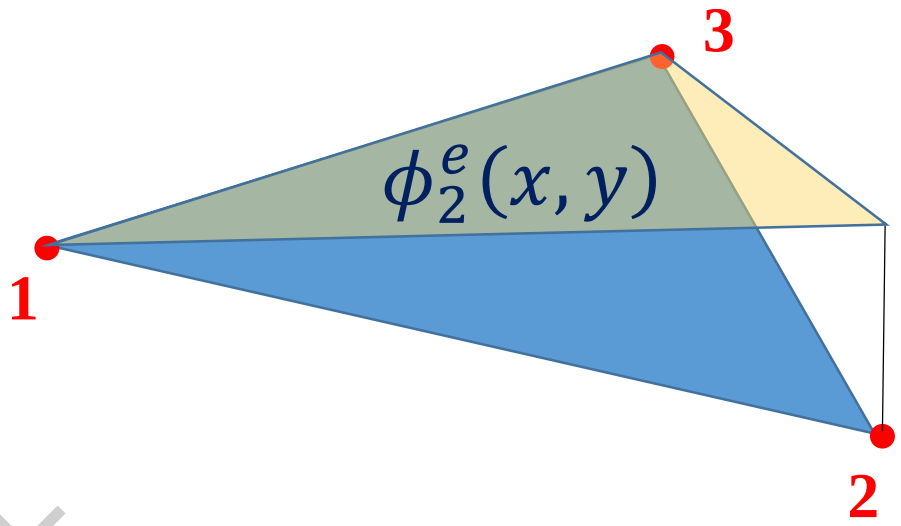
For the given problem $f = 9\text{W/m}^2$

$$\{f^e\} = f \int_{\Omega^e} \phi_i^e \, dxdy$$

NPTEL

Evaluation of boundary integral





$$\int_{\Omega^e} \phi_i^e \, dx dy = \frac{A^e}{3}$$

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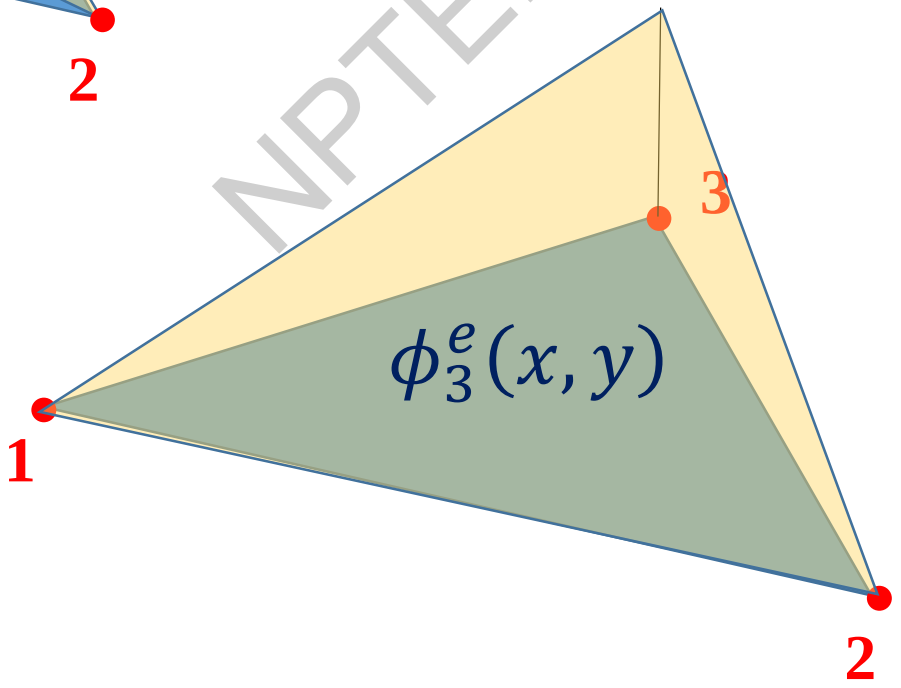
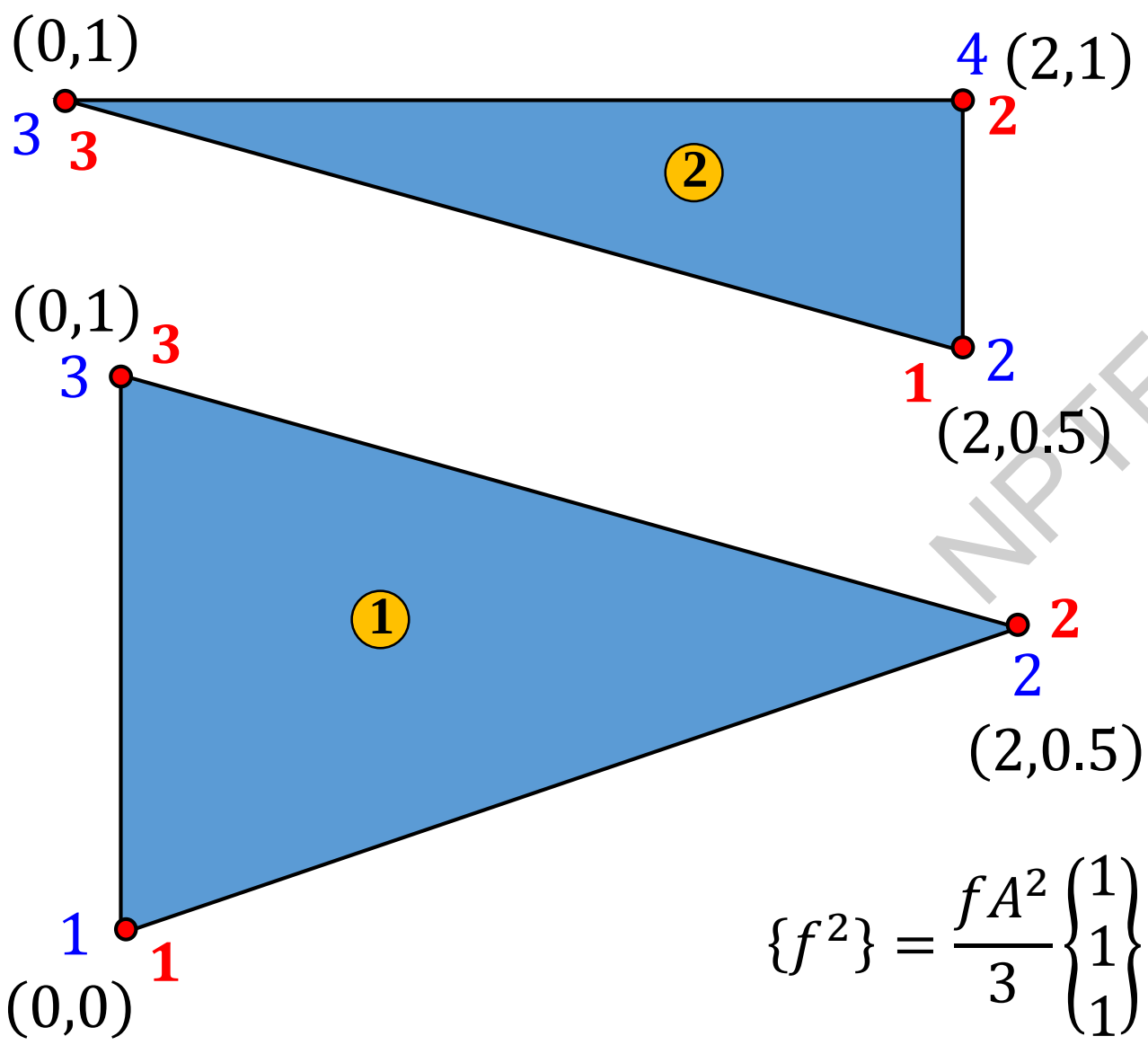


Step 2: Element Equation (Source Term $\{f^e\}$)



$$\{f^e\} = \int_{\Omega^e} f \phi_i^e \, dxdy$$

For the given problem $f = 9\text{W}/m^2$

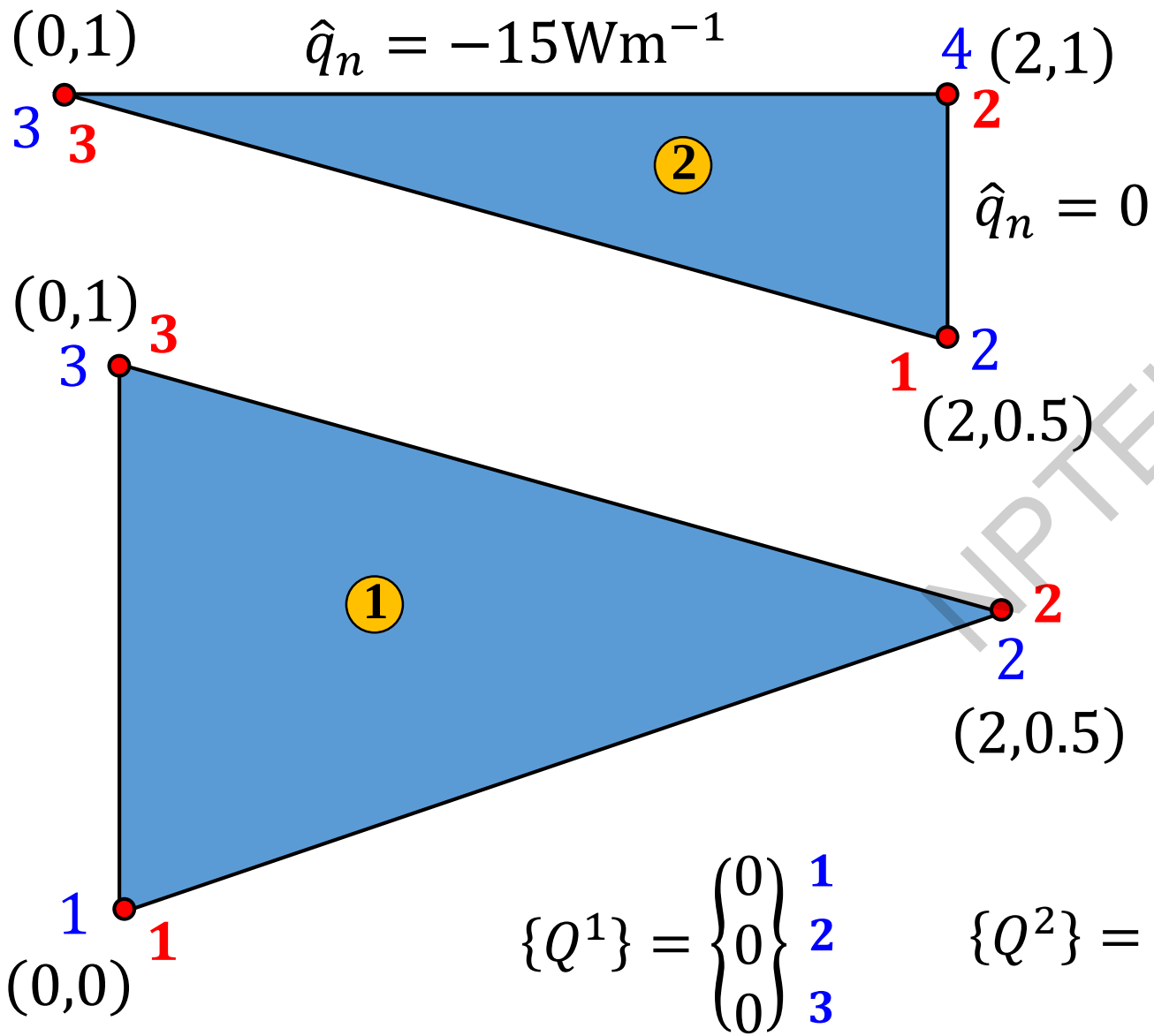
$$\{f^e\} = f \int_{\Omega^e} \phi_i^e \, dxdy = \frac{fA^e}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$$

$$\{f^1\} = \frac{fA^1}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix} = \begin{Bmatrix}3\\3\\3\end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

$$\{f^2\} = \frac{fA^2}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix} = \begin{Bmatrix}1.5\\1.5\\1.5\end{Bmatrix}\begin{matrix}2\\4\\3\end{matrix}$$

NPTEL

Step 2: Element Equation (Boundary Flux Term $\{Q^e\}$)

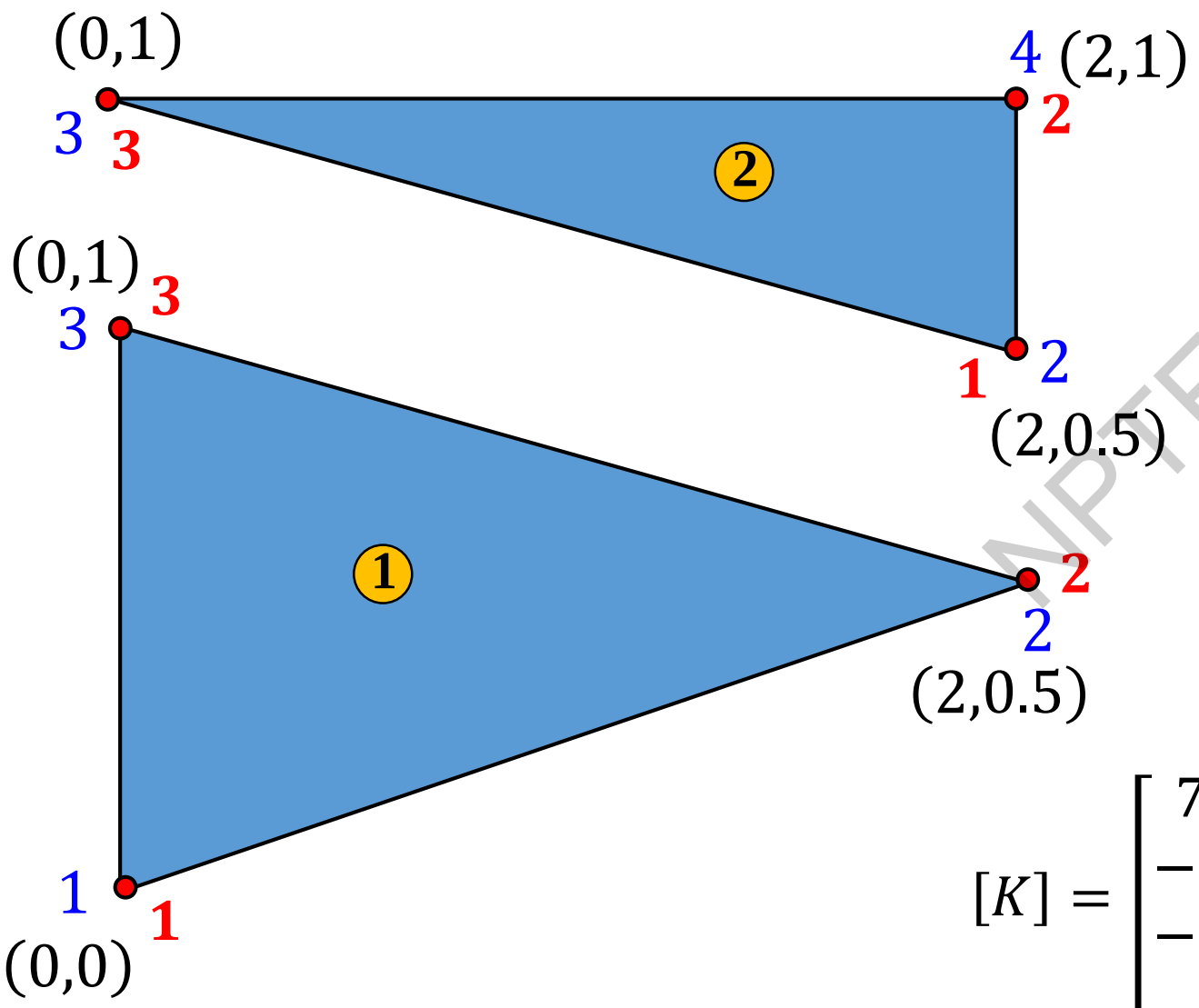


$$\{Q^e\} = \oint_{\Gamma^e} \phi_i^e q_n \, ds$$

$$\{Q^1\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \qquad \{Q^2\} = \begin{Bmatrix} 0 \\ -15 \\ -15 \end{Bmatrix} \begin{matrix} 2 \\ 4 \\ 3 \end{matrix}$$

NPTEL

Step 3: Assembly of Element Equation



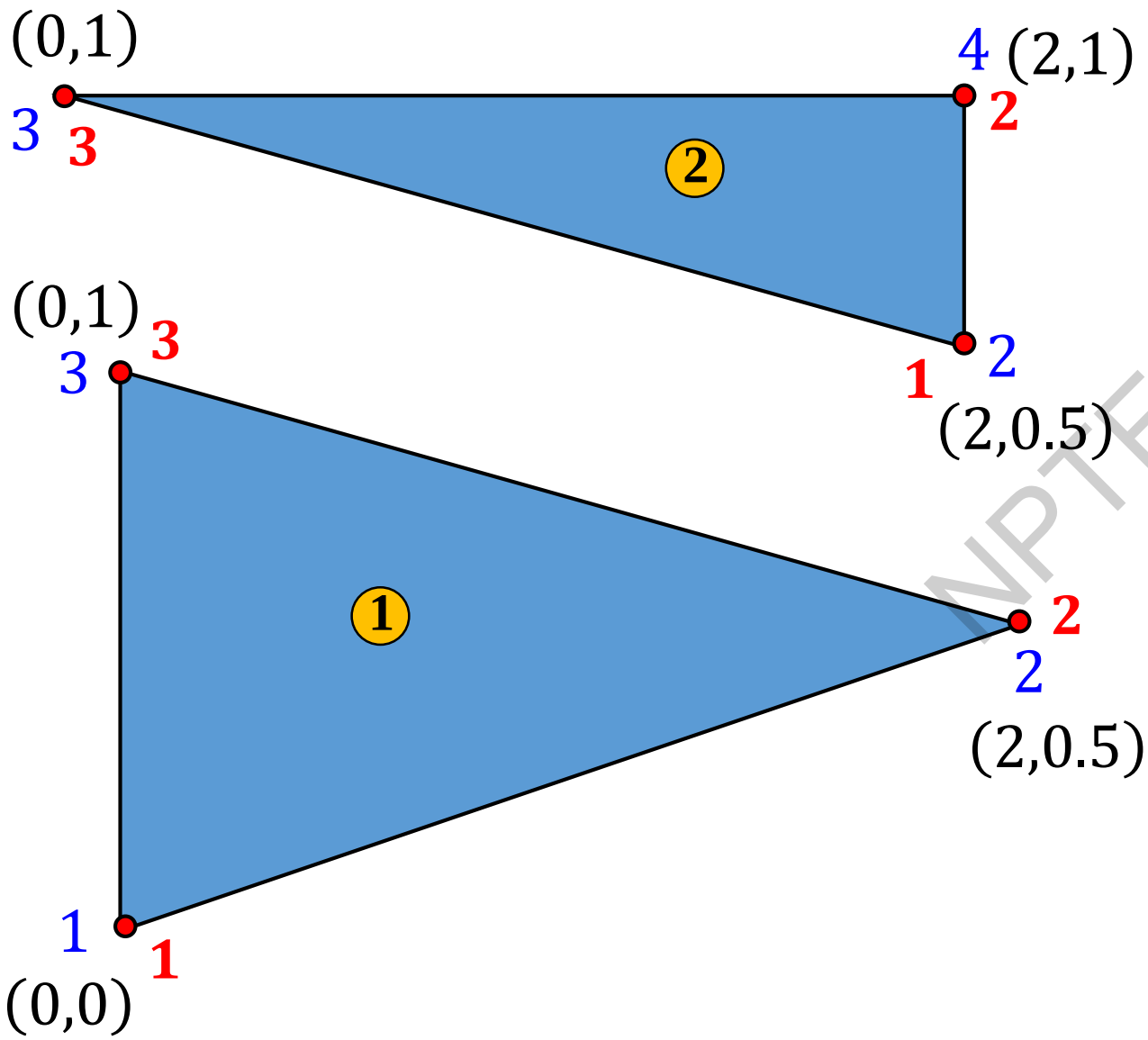
$$[K^1] = \begin{matrix} \\ 1 \\ 2 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 7.4375 & -0.8750 & -6.5625 \\ -0.8750 & 1.7500 & -0.8750 \\ -6.5625 & -0.8750 & 7.4375 \end{bmatrix} \end{matrix}$$

$$[K^2] = \begin{matrix} \\ 2 \\ 4 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 2 & 4 & 3 \end{matrix} \\ \begin{bmatrix} 14 & -14 & 0 \\ -14 & 14.8750 & -0.8750 \\ 0 & -0.8750 & 0.8750 \end{bmatrix} \end{matrix}$$

$$[K] = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 7.4375 & -0.8750 & -6.5625 & 0 \\ -0.8750 & 15.7500 & -0.8750 & -14 \\ -6.5625 & -0.8750 & 8.3125 & -0.8750 \\ 0 & -14 & -0.8750 & 14.8750 \end{bmatrix} \end{matrix}$$

NPTEL

Step 3: Assembly of Element Equation



$$\{f^1\} = \begin{Bmatrix} 3 \\ 3 \\ 3 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \qquad \{f^2\} = \begin{Bmatrix} 1.5 \\ 1.5 \\ 1.5 \end{Bmatrix} \begin{matrix} 2 \\ 4 \\ 3 \end{matrix}$$

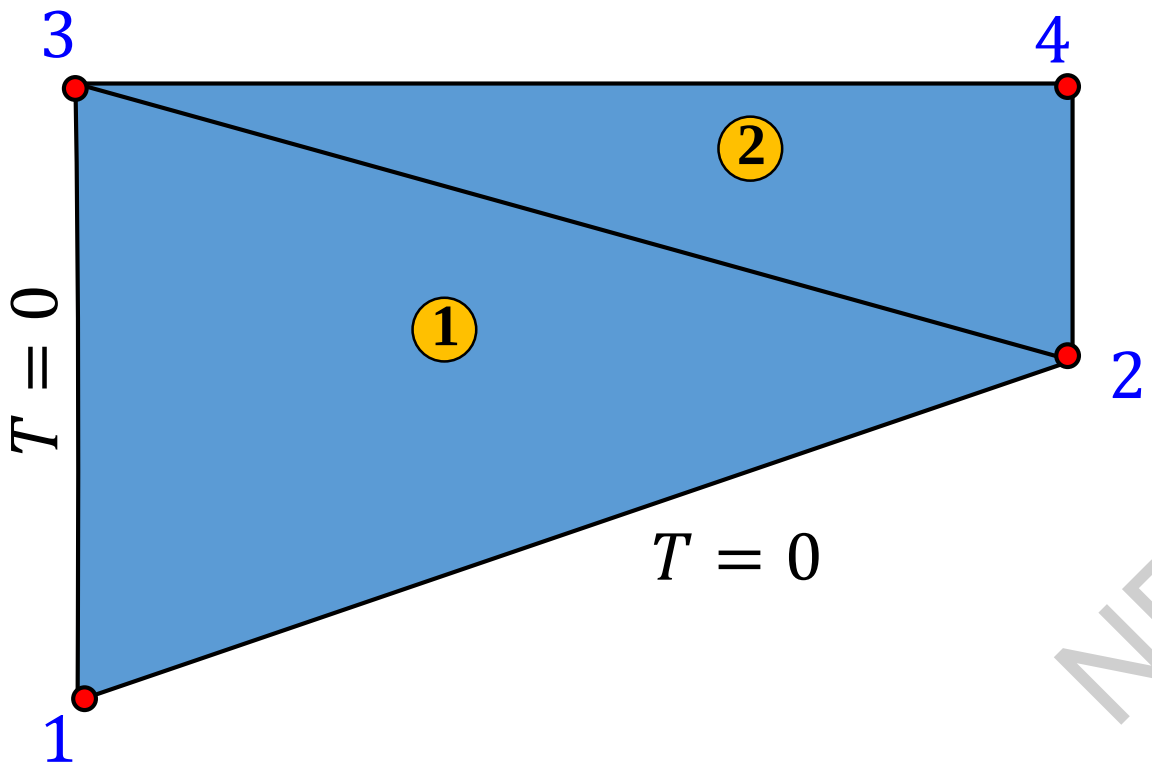
$$\{f\} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4.5 \\ 4.5 \\ 1.5 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\{Q^1\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \qquad \{Q^2\} = \begin{Bmatrix} 0 \\ -15 \\ -15 \end{Bmatrix} \begin{matrix} 2 \\ 4 \\ 3 \end{matrix}$$

$$\{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -15 \\ -15 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

NPTEL

Step 3: Assembly of Element Equation



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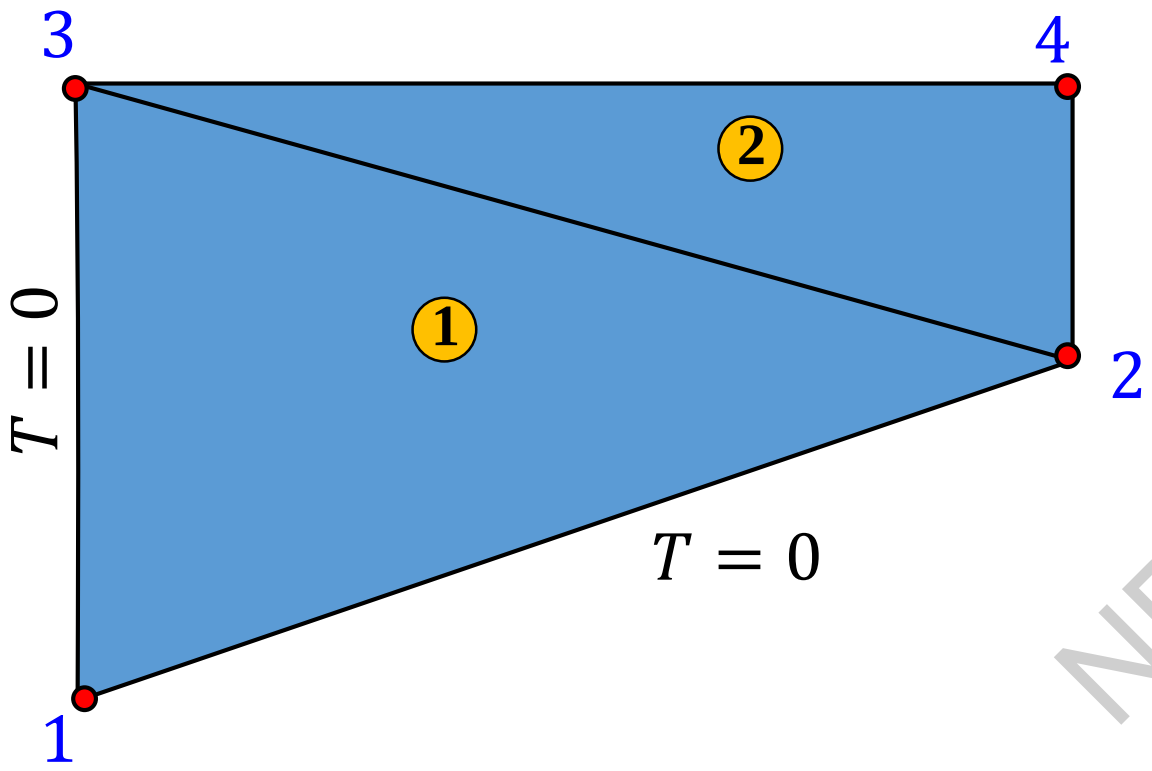
$$\begin{bmatrix} 7.4375 & -0.8750 & -6.5625 & 0 \\ -0.8750 & 15.7500 & -0.8750 & -14 \\ -6.5625 & -0.8750 & 8.3125 & -0.8750 \\ 0 & -14 & -0.8750 & 14.8750 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4.5 \\ 4.5 \\ 1.5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -15 \\ -15 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \end{Bmatrix}$$

Step 4: Imposition of Boundary Conditions

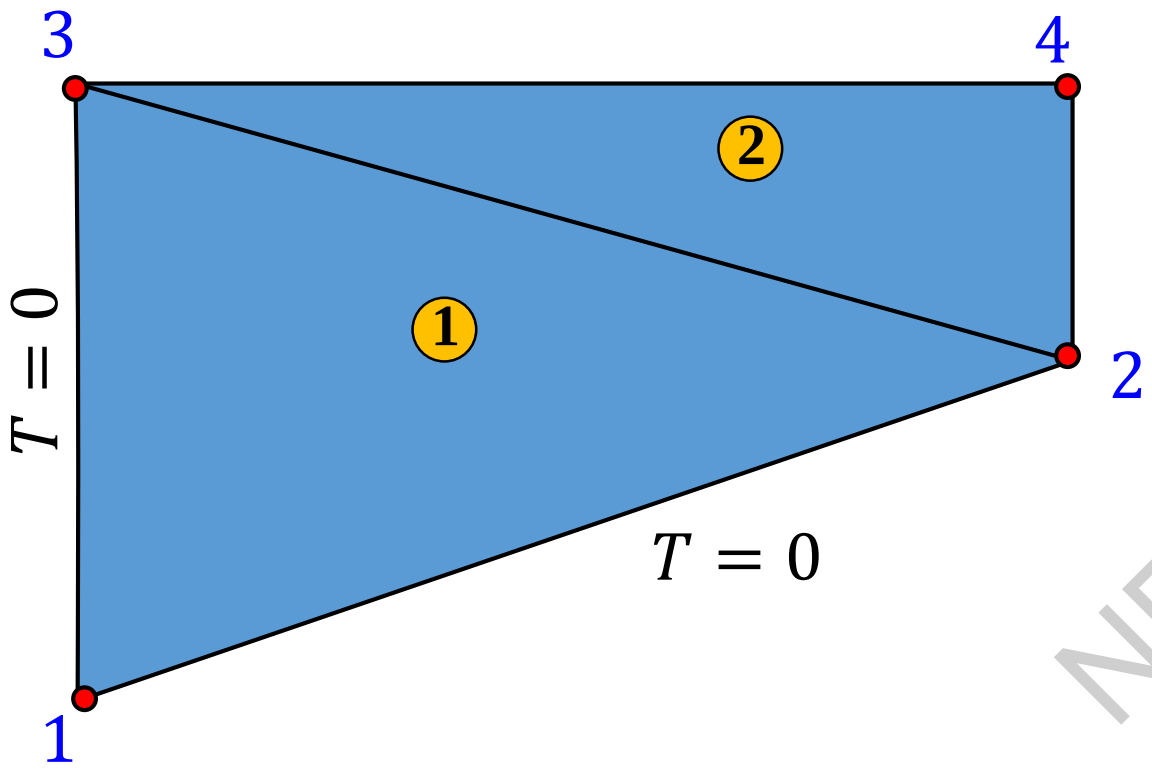


NPTEL

$$\begin{bmatrix} 7.4375 & -0.8750 & -6.5625 & 0 \\ -0.8750 & 15.7500 & -0.8750 & -14 \\ -6.5625 & -0.8750 & 8.3125 & -0.8750 \\ 0 & -14 & -0.8750 & 14.8750 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4.5 \\ -11.5 \\ -13.5 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \end{Bmatrix}$$

$T_1 = 0$, $T_2 = 0$, $T_3 = 0$

Step 5: Solutions



$$T_4 = -\frac{13.5}{14.875} = -0.90756$$

NPTEL

$$\left[\begin{array}{ccc:c} 7.4375 & -0.8750 & -6.5625 & 0 \\ -0.8750 & 15.7500 & -0.8750 & -14 \\ -6.5625 & -0.8750 & 8.3125 & -0.8750 \\ \hdashline 0 & -14 & -0.8750 & 14.8750 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4.5 \\ -11.5 \\ -13.5 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \end{Bmatrix}$$

NPTEL

THANK YOU !

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Finite Element Method

Week 8: Finite element formulation for 2D scalar field problems

Lecture 39: Iso-parametric Formulation

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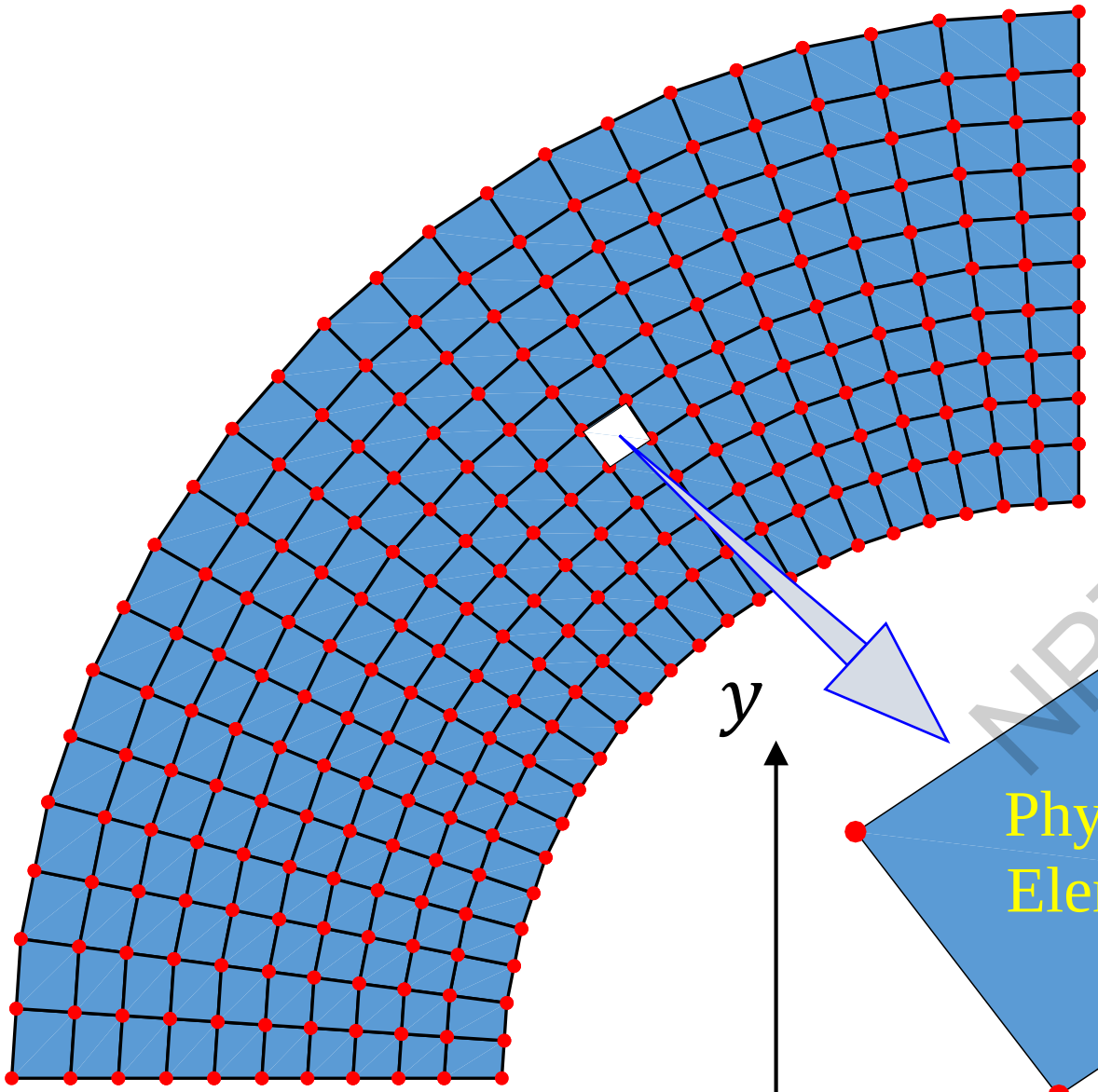


Iso-parametric formulation

η

ξ

Parent Element

Parametric domain
$[-1,1] \times [-1,1]$
(ξ, η) natural coordinate

y

x

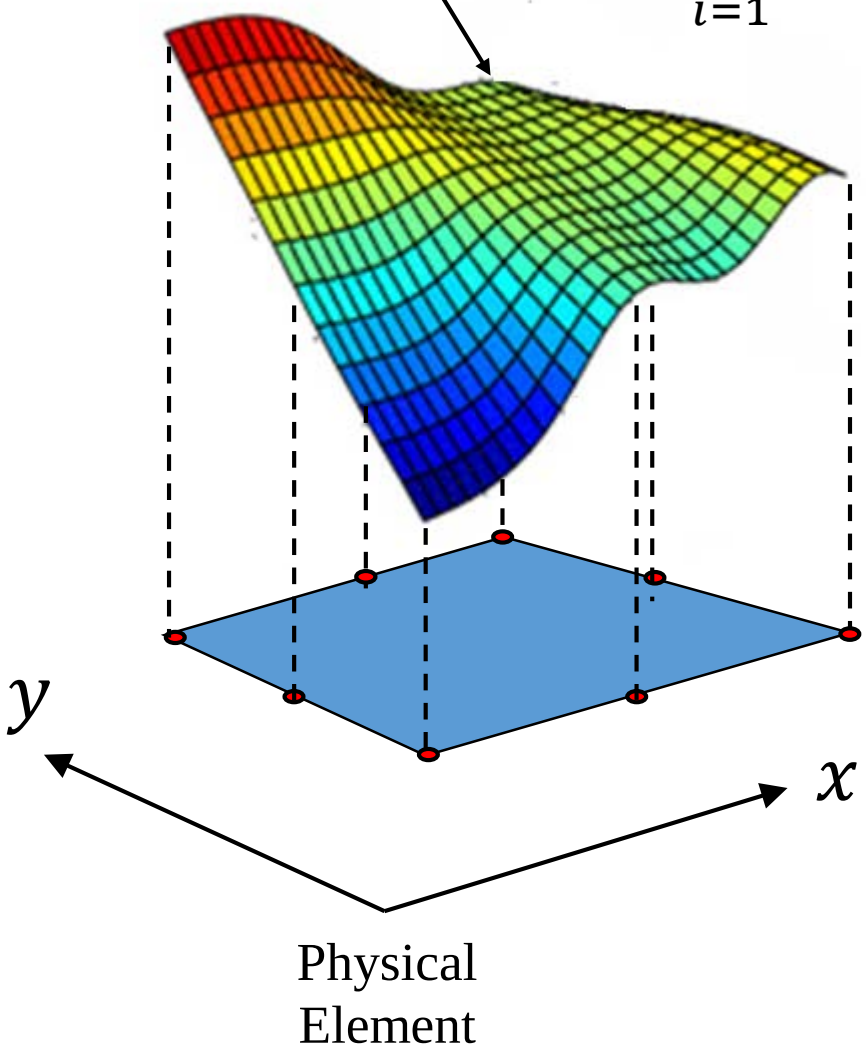
Physical Element

Physical domain

NPTEL

Iso-parametric Formulation: Concept

$$u(x^e, y^e) = \sum_{i=1}^{n} \phi_i^e u_i^e$$



Physical Element

Iso — **Equal/ Similar**

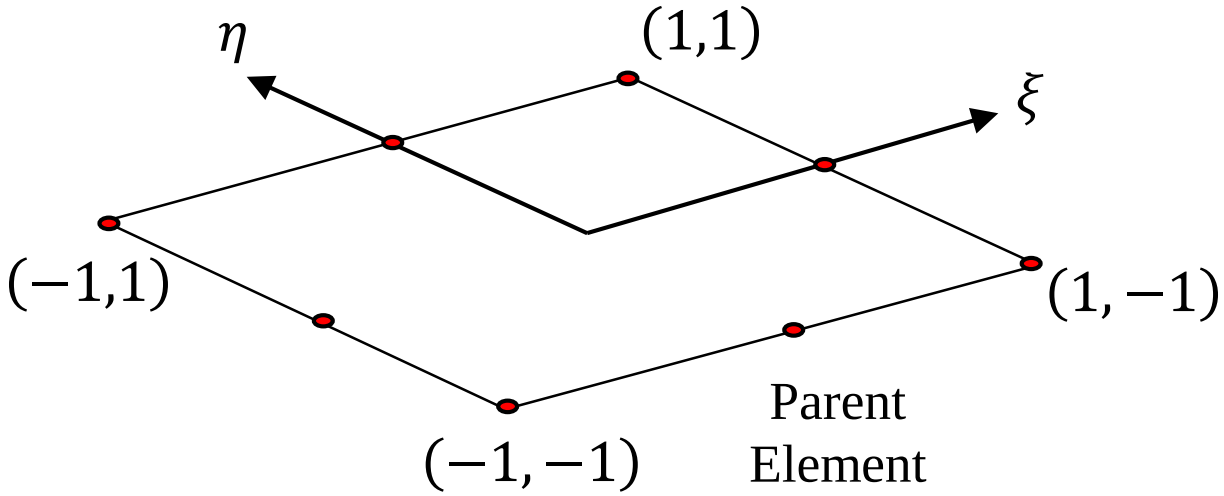
Parametric — **Expressing in terms of parameters**

Expressing What?

Ans: Field variables and geometry

Iso-parametric

Expressing in terms of similar parameters

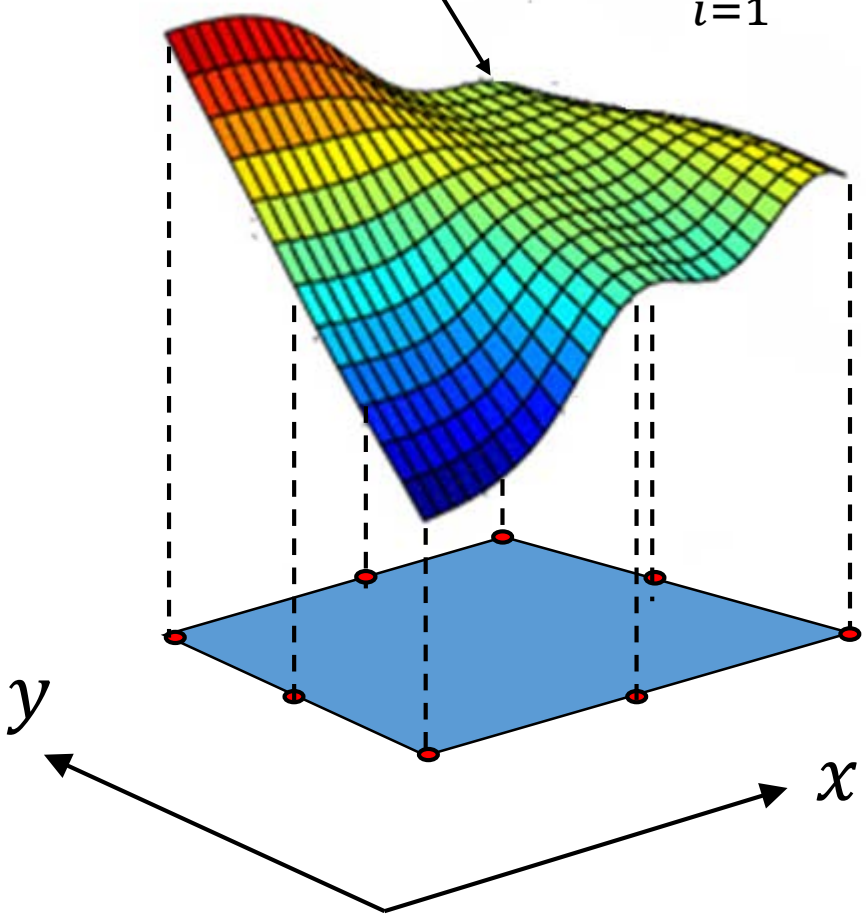


Parent Element

NPTEL

Iso-parametric Formulation: Concept

$$u(x^e, y^e) = \sum_{i=1}^{n} \phi_i^e u_i^e$$



Physical Element

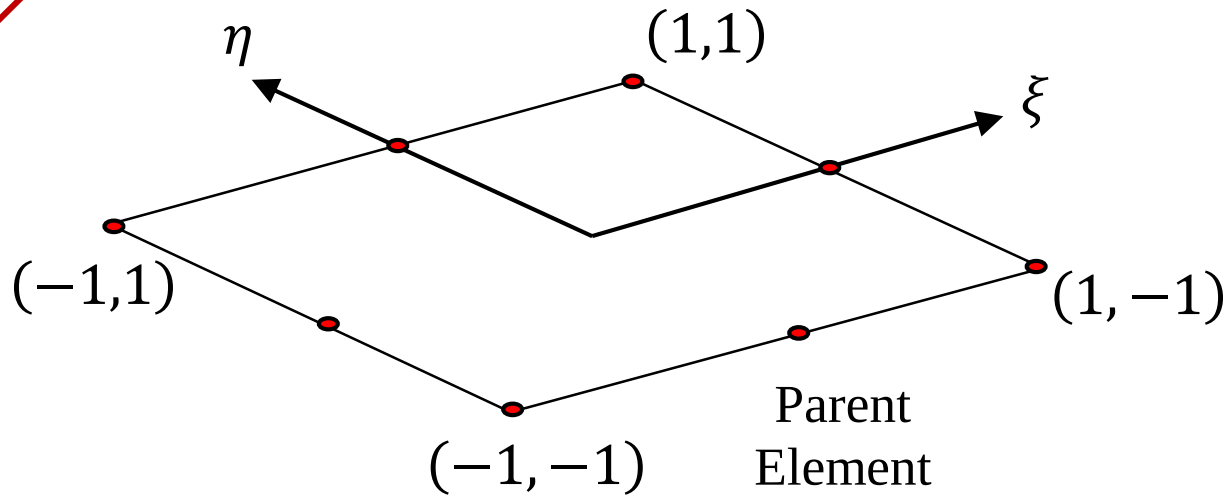
Iso-parametric

Expressing in terms of similar parameters

$$x^e = \sum_{i=1}^{n} \phi_i(\xi, \eta) x_i^e$$

$$y^e = \sum_{i=1}^{n} \phi_i(\xi, \eta) y_i^e$$

NPTEL



Parent Element

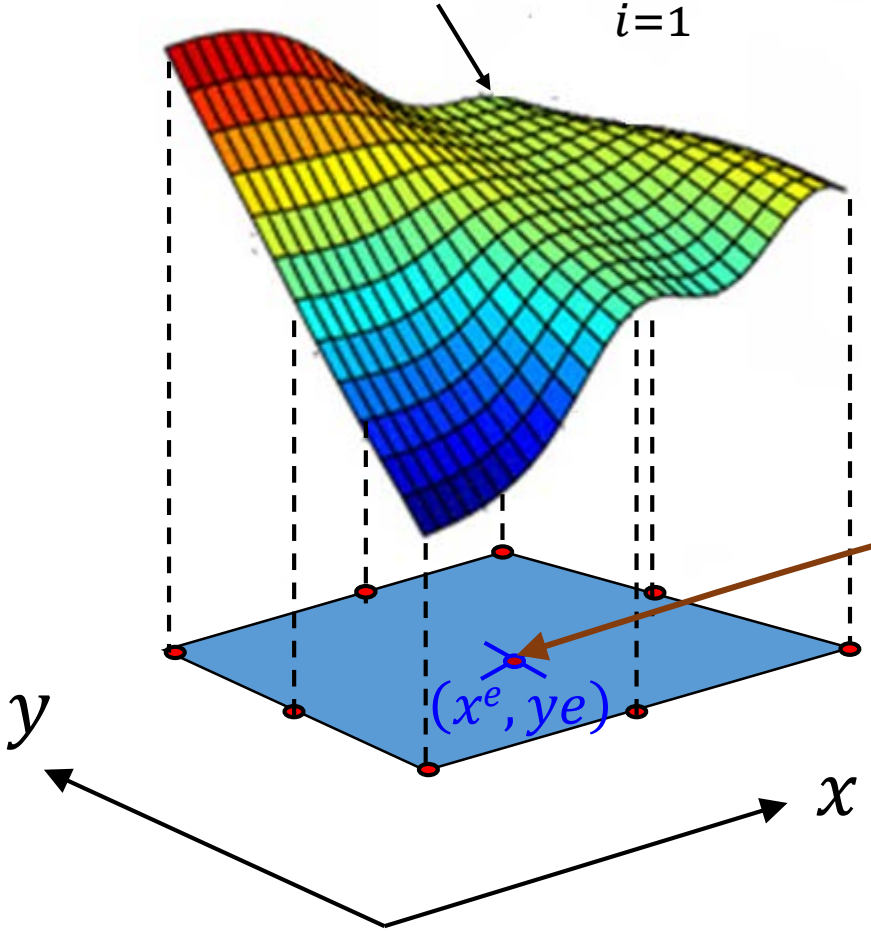
Iso-parametric Formulation: Concept

$$u(x^e, y^e) = \sum_{i=1}^{n} \phi_i^e(\xi, \eta) u_i^e$$

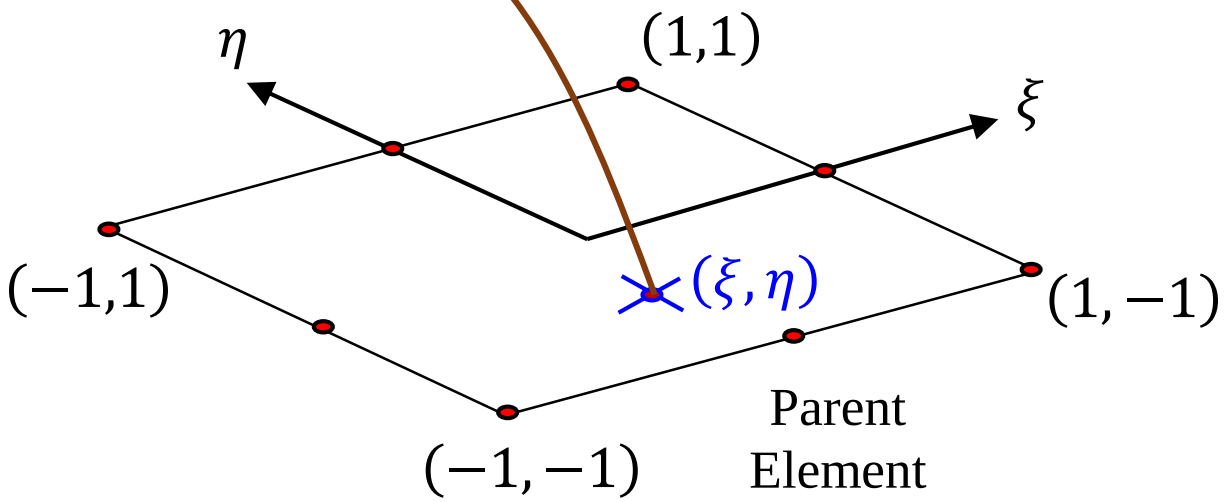
Iso-parametric

Expressing in terms of similar parameters

$$x^e = \sum_{i=1}^{n} \phi_i(\xi, \eta) x_i^e \qquad y^e = \sum_{i=1}^{n} \phi_i(\xi, \eta) y_i^e$$

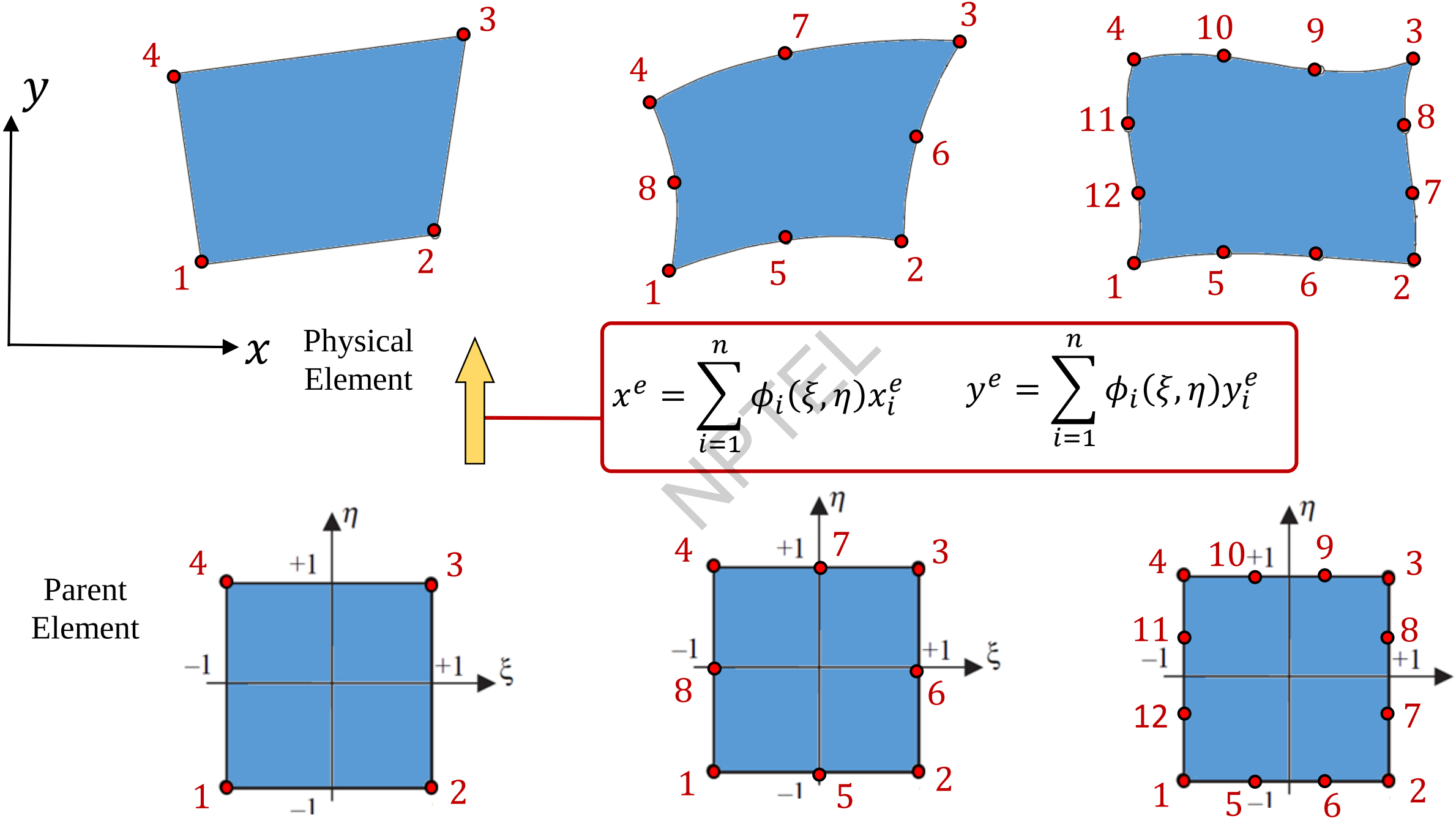


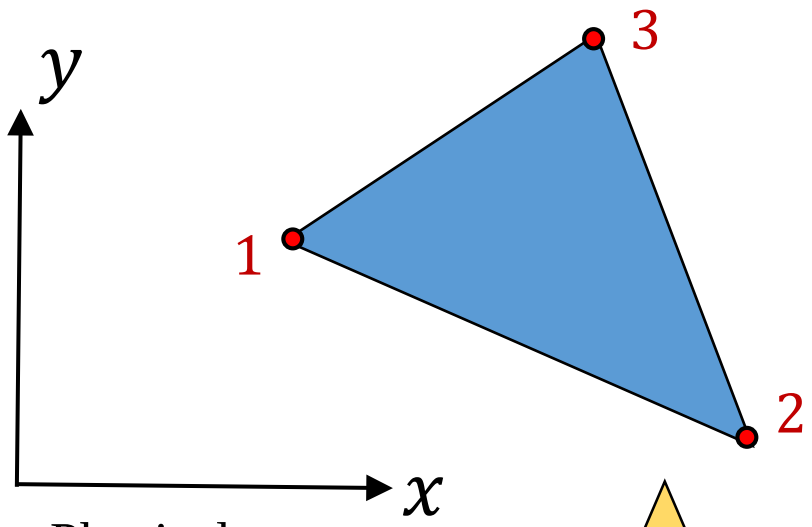
Physical Element



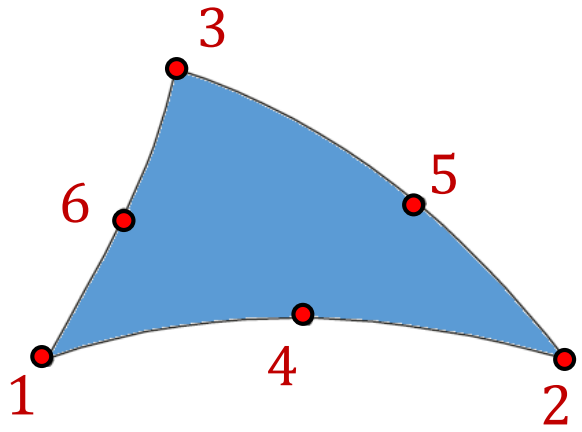
Parent Element

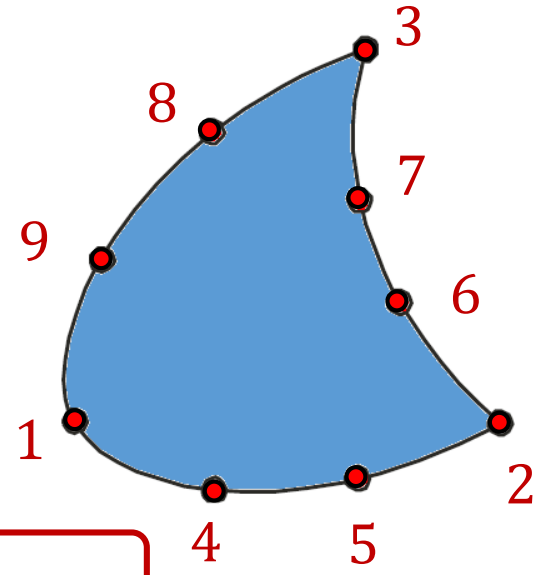
NPTEL


y
x
4
3
1
2
7
3
4
6
8
5
1
2
4
10
9
3
11
8
12
7
1
5
6
2
Physical Element
$x^e = \sum_{i=1}^{n} \phi_i(\xi,\eta) x_i^e \qquad y^e = \sum_{i=1}^{n} \phi_i(\xi,\eta) y_i^e$
Parent Element
η
ξ
+1
−1
4
3
1
2
7
8
5
6
10
9
11
12
NPTEL



Physical Element

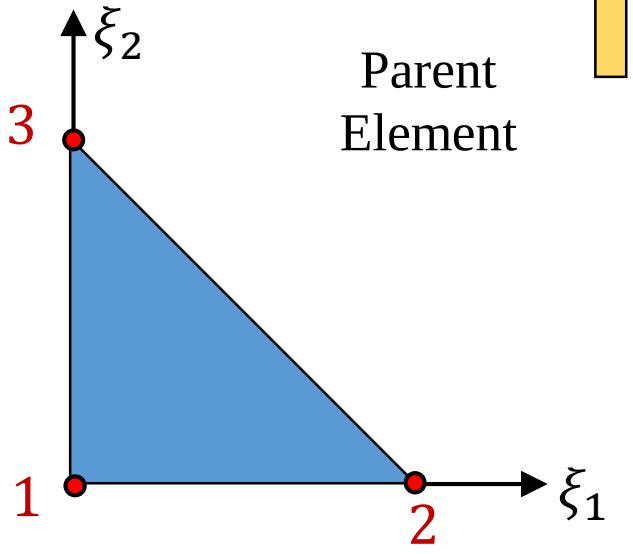


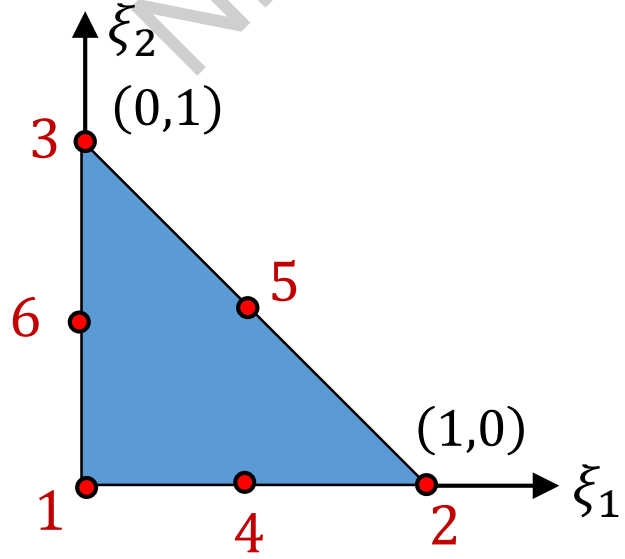


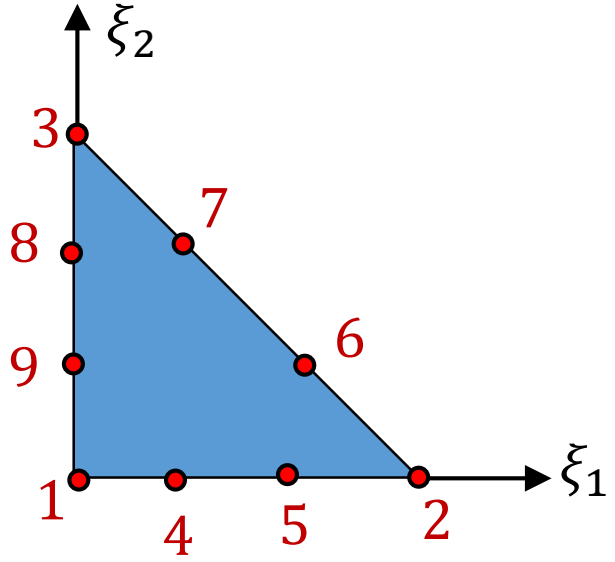
$$x^e = \sum_{i=1}^{n} \phi_i(\xi,\eta) x_i^e \qquad y^e = \sum_{i=1}^{n} \phi_i(\xi,\eta) y_i^e$$

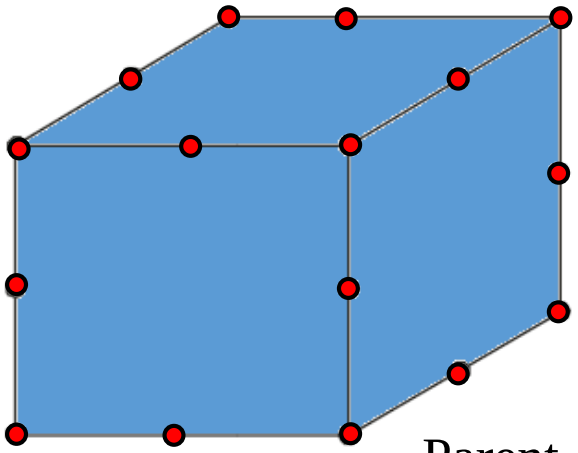
NPTEL

Parent Element









Parent Element

$$x^e = \sum_{i=1}^{n} \phi_i(\xi, \eta, \zeta) x_i^e$$

$$y^e = \sum_{i=1}^{n} \phi_i(\xi, \eta, \zeta) y_i^e$$

$$z^e = \sum_{i=1}^{n} \phi_i(\xi, \eta, \zeta) z_i^e$$

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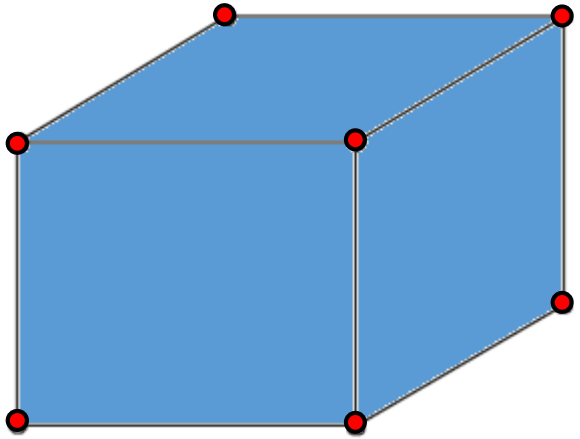
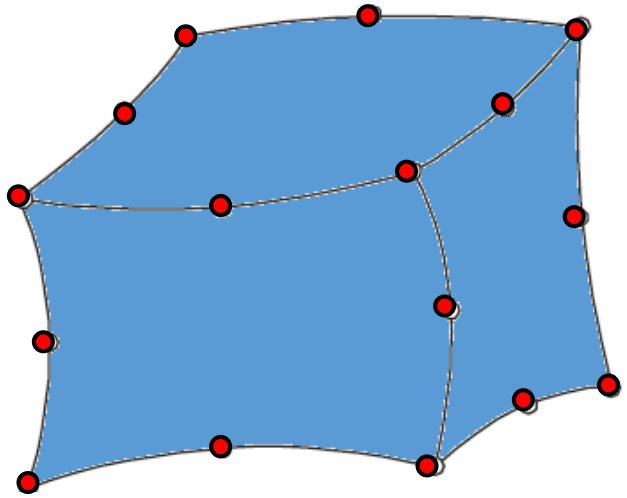
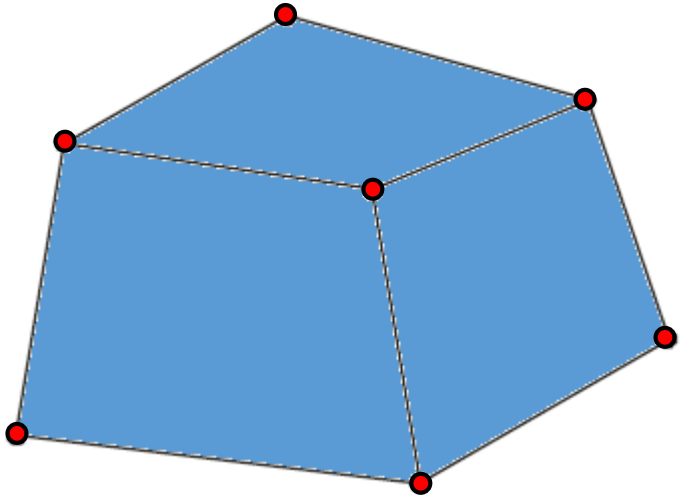
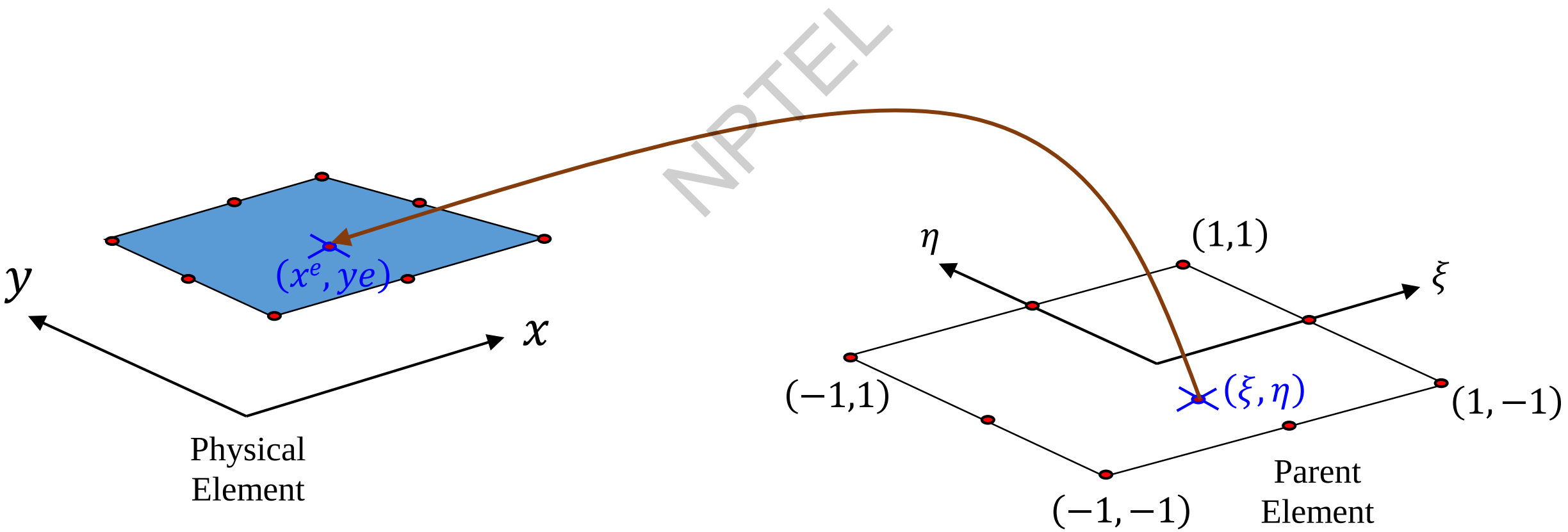




Physical Element



Uniqueness



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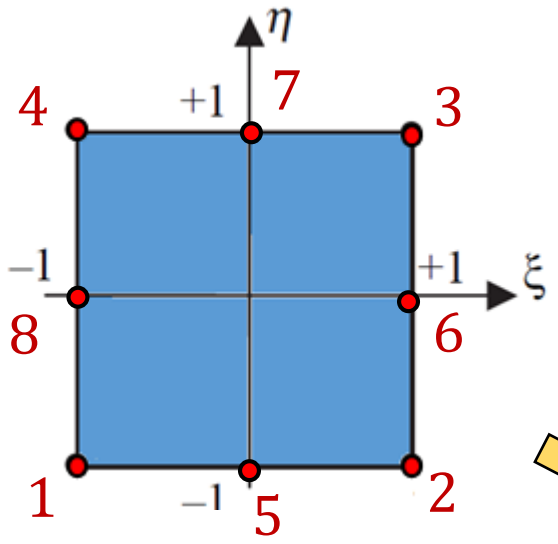
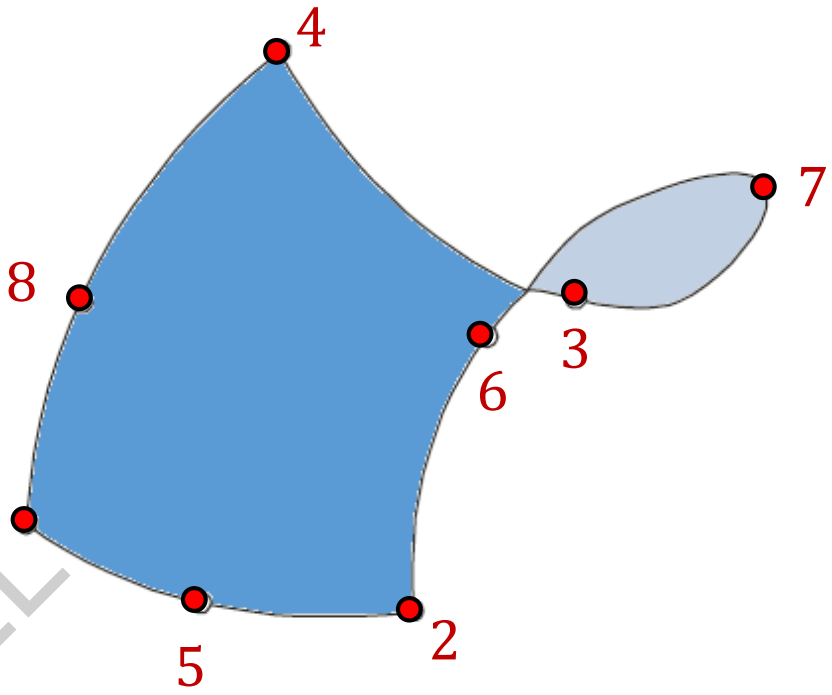
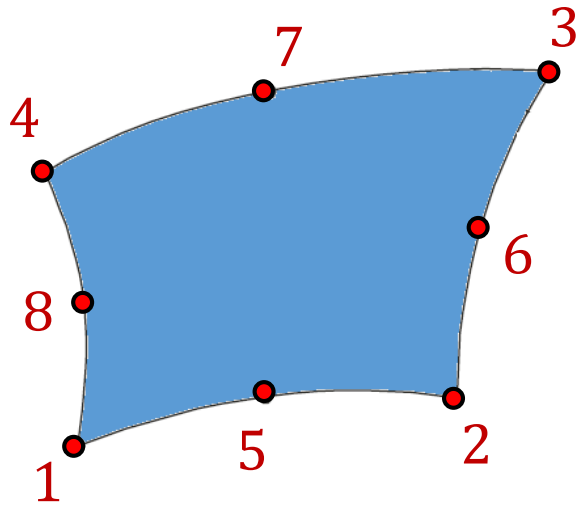
Uniqueness







NPTEL

Recall: Poisson equation

$$\sum_{j=1}^{n}\left\{\int_{\Omega^e}\left[\frac{\partial\phi_i^e}{\partial x}\left(k\frac{\partial\phi_j^e}{\partial x}\right)+\frac{\partial\phi_i^e}{\partial y}\left(k\frac{\partial\phi_j^e}{\partial y}\right)\right]dxdy\right\}u_j^e=\int_{\Omega^e}f\phi_i^e\,dxdy+\oint_{\Gamma^e}\phi_i^e q_n\,ds$$

$$[K^e]\{u^e\}=\{f^e\}+\{Q^e\}$$

$$[K^e]=\int_{\Omega^e}\mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e\,dxdy \qquad \mathbf{C}=\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$$

$$\mathbf{B}^e=\begin{bmatrix}\frac{\partial\phi_1^e}{\partial x} & \frac{\partial\phi_2^e}{\partial x} & \cdots & \frac{\partial\phi_n^e}{\partial x}\\ \frac{\partial\phi_1^e}{\partial y} & \frac{\partial\phi_2^e}{\partial y} & \cdots & \frac{\partial\phi_n^e}{\partial y}\end{bmatrix}$$

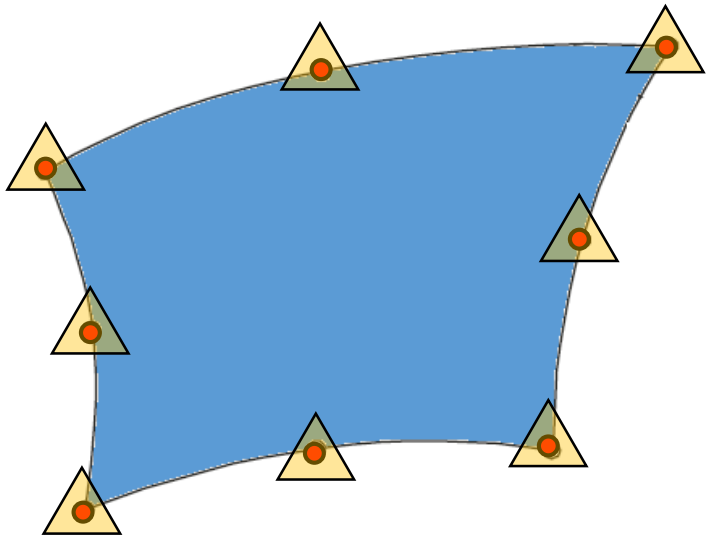
n = number of nodes

NPTEL

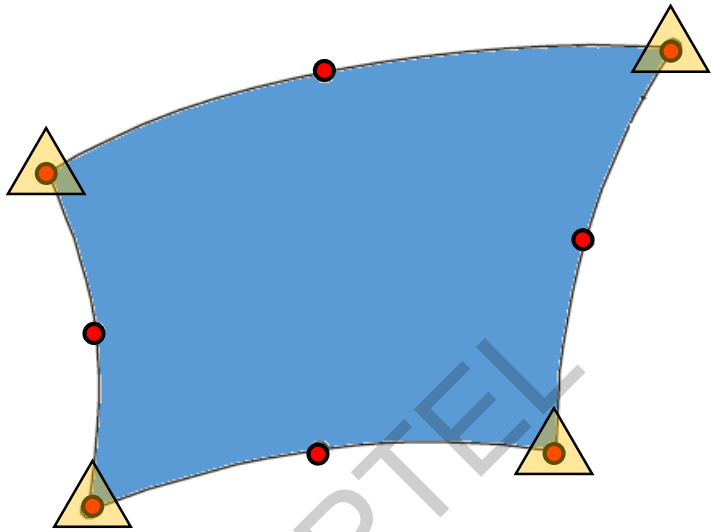
Iso-parametric, Super-parametric and Sub-parametric

NPTEL

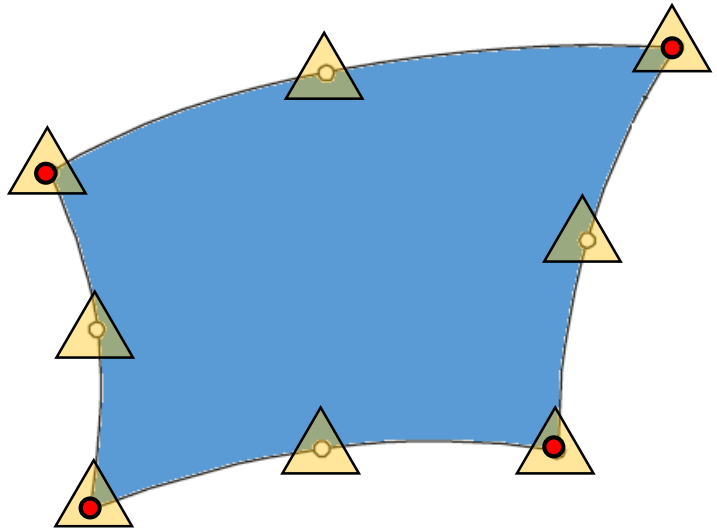
Iso-parametric, Super-parametric and Sub-parametric



Iso-parametric



Super-parametric



Sub-parametric

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- Nodes for approximating geometry

△ Nodes for approximating field variables

NPTEL

THANK YOU !

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Finite Element Method

Week 8: Finite element formulation for 2D scalar field problems

Lecture 40: Example with Quadrilateral element

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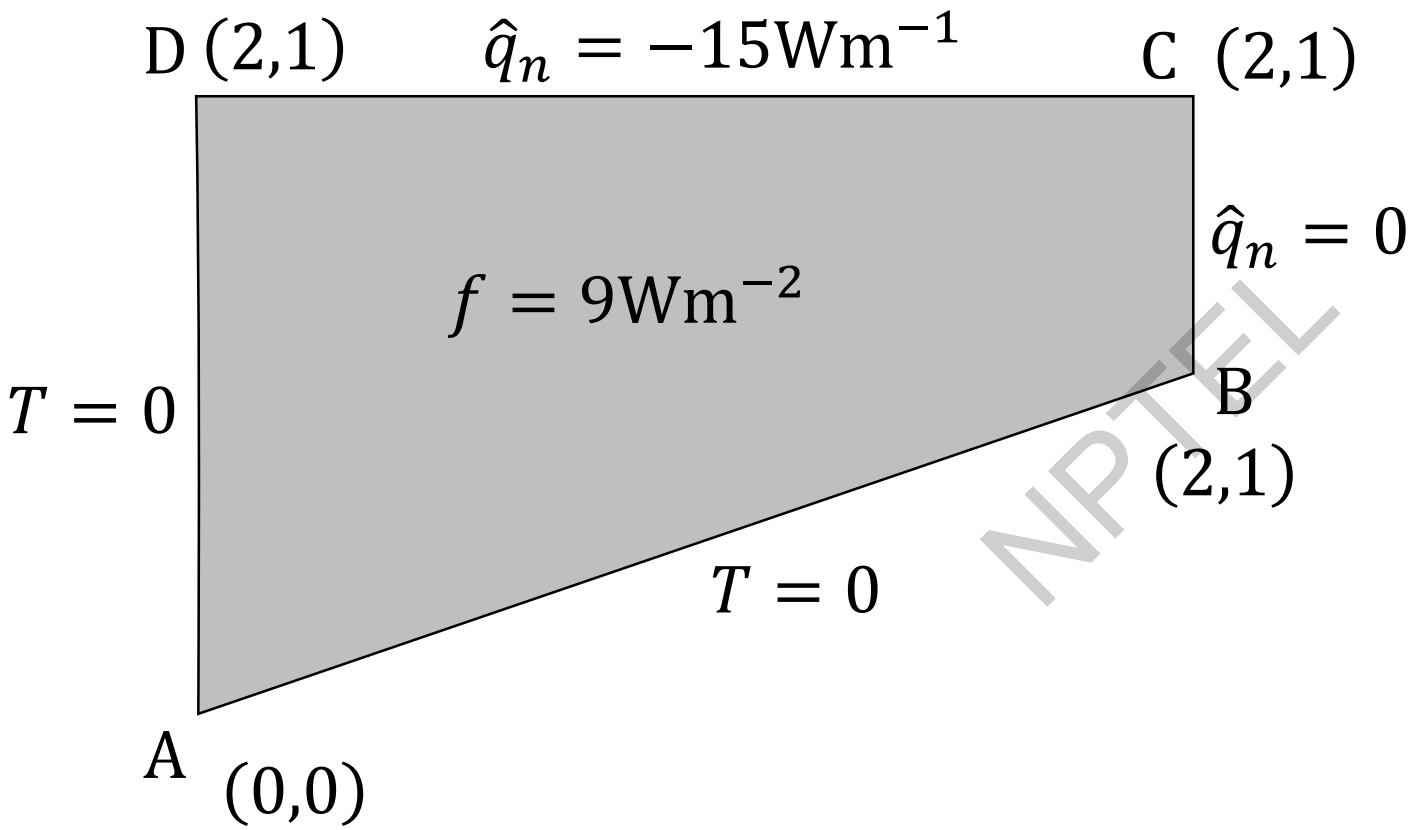
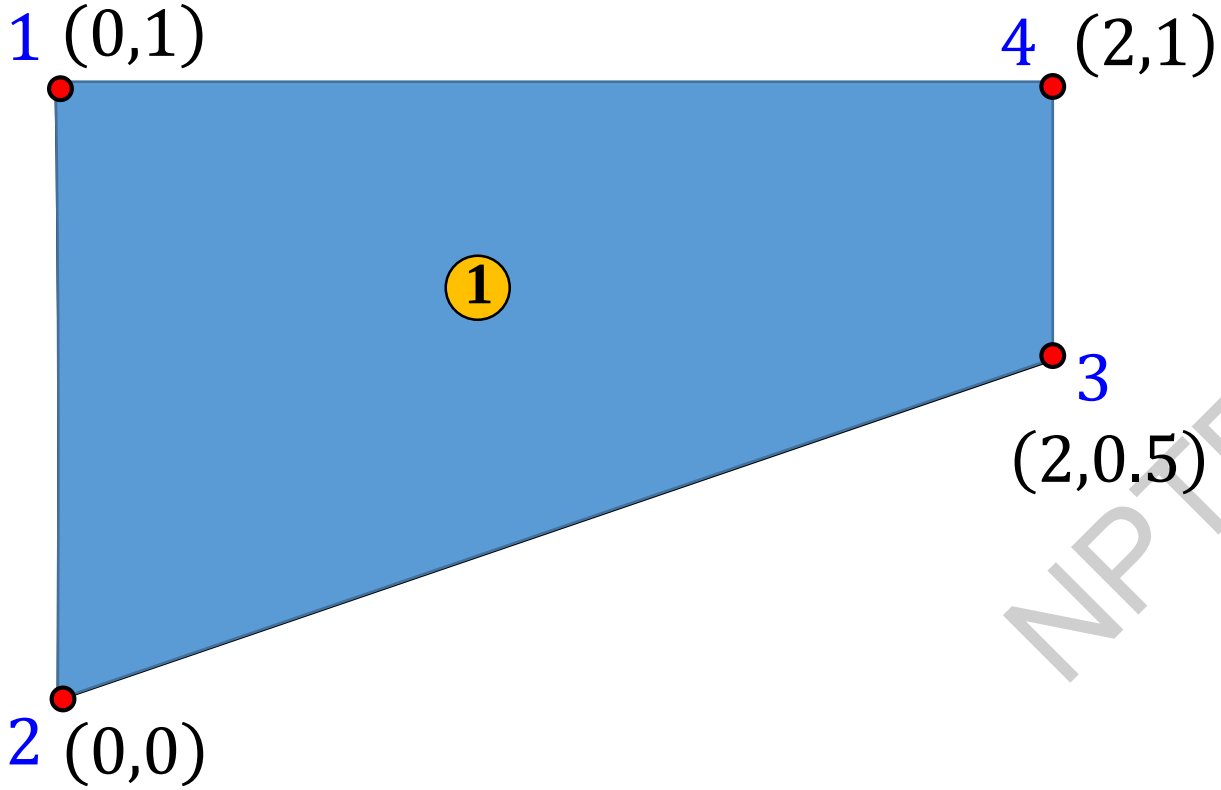
Amit Shaw
Associate Professor
Department of Civil Engineering
IIT Kharagpur

Example 1: Heat Conduction Problem $\nabla \cdot (k\nabla T) = -f$



NPTEL

Step 1: Discretization



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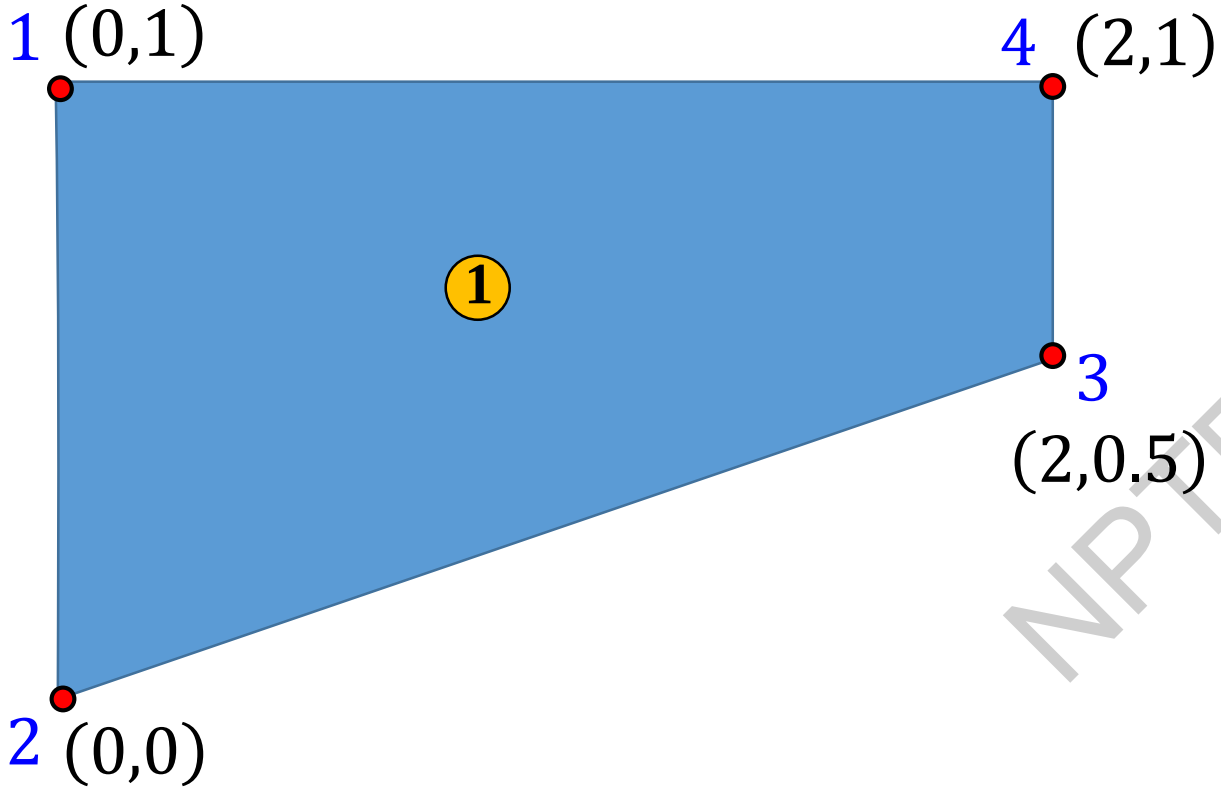
Step 2: Element Equation



$$[K^e]\{T^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{e\mathrm{T}} \mathbf{C} \mathbf{B}^e \, dxdy$$

$$\{f^e\} = \int_{\Omega^e} f^e \phi_i^e \, dxdy$$

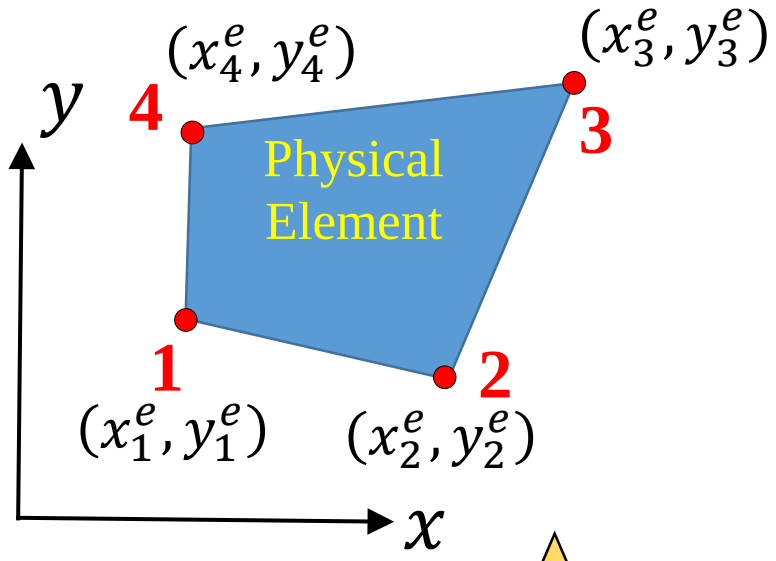
$$\{Q^e\} = \oint_{\Gamma^e} \phi_i^e q_n^e \, ds$$

$$\mathbf{B}^e = \begin{bmatrix} \dfrac{\partial \phi_1^e(x,y)}{\partial x} & \dfrac{\partial \phi_2^e(x,y)}{\partial x} & \dfrac{\partial \phi_3^e(x,y)}{\partial x} & \dfrac{\partial \phi_4^e(x,y)}{\partial x} \\ \dfrac{\partial \phi_1^e(x,y)}{\partial y} & \dfrac{\partial \phi_2^e(x,y)}{\partial y} & \dfrac{\partial \phi_3^e(x,y)}{\partial y} & \dfrac{\partial \phi_4^e(x,y)}{\partial y} \end{bmatrix}$$

$$\mathbf{C} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

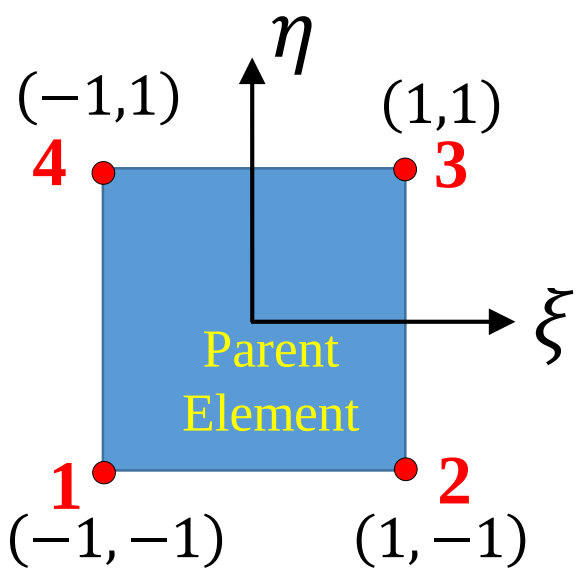
NPTEL

Recall: Isoparametric Formulation



$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [\mathrm{J}]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$[\mathrm{J}] = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} x_i^e & \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} y_i^e \\ \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} x_i^e & \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} y_i^e \end{bmatrix}$$



$$x^e = \sum_{i=1}^{4} \phi_i(\xi,\eta) x_i^e$$

$$y^e = \sum_{i=1}^{4} \phi_i(\xi,\eta) y_i^e$$

$$\phi_1(\xi,\eta) = (1-\xi)(1-\eta)/4$$

$$\phi_2(\xi,\eta) = (1+\xi)(1-\eta)/4$$

$$\phi_3(\xi,\eta) = (1+\xi)(1+\eta)/4$$

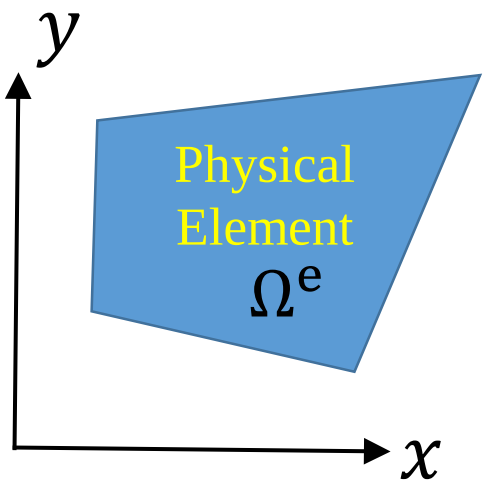
$$\phi_4(\xi,\eta) = (1-\xi)(1+\eta)/4$$

NPTEL

Recall: Integration Over Quadrilateral Domain

$$I = \int_{\Omega^e} f(x,y)\,d\Omega \Rightarrow I = \int_{\eta=-1}^{1}\int_{\xi=-1}^{1} |\mathbf{J}(\xi,\eta)| f(\xi,\eta)\,d\xi\,d\eta$$

$$\Rightarrow I = \sum_{i=1}^{n_{gp}}\sum_{j=1}^{n_{gp}} W_i W_j \left|\mathbf{J}\left(\xi_i,\eta_j\right)\right| f\left(\xi_i,\eta_j\right)$$



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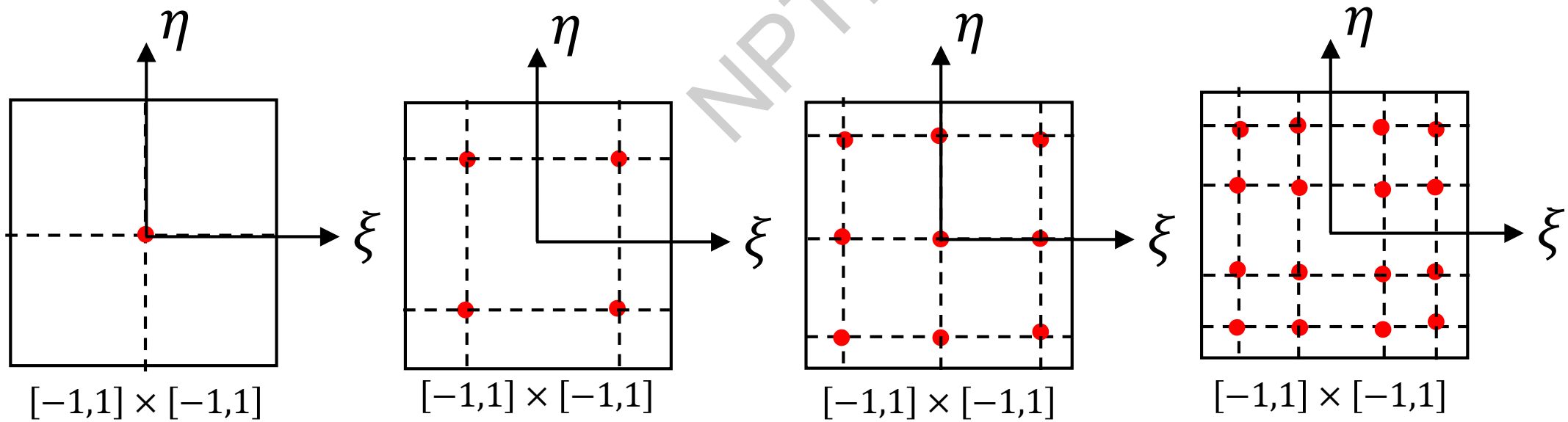
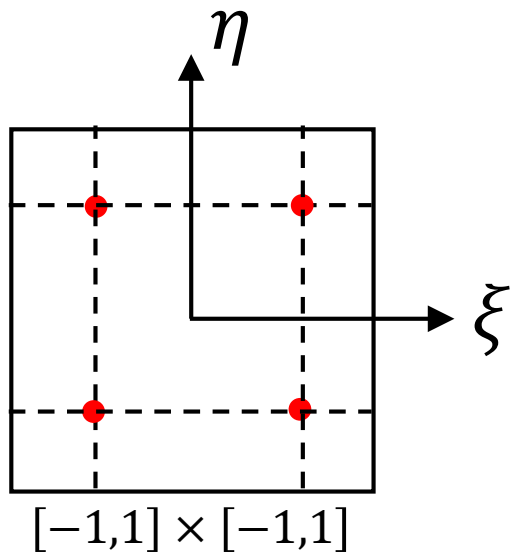


Step 2: Element Equation (Coefficient Matrix $[K^e]$)

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{e\mathrm{T}} \mathbf{C} \mathbf{B}^e \, dx dy \qquad \mathbf{C} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow [K^e] = \int_{\eta=-1}^{1} \int_{\xi=-1}^{1} |\mathbf{J}(\xi,\eta)| \mathbf{B}^{e\mathrm{T}}(\xi,\eta) \mathbf{C} \mathbf{B}^e(\xi,\eta) \, d\xi d\eta$$

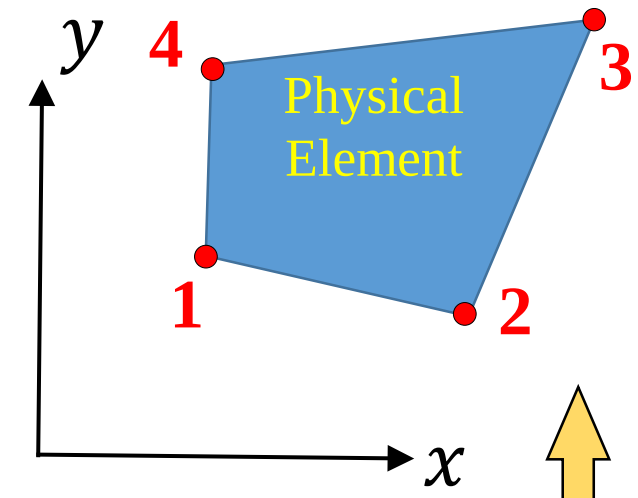


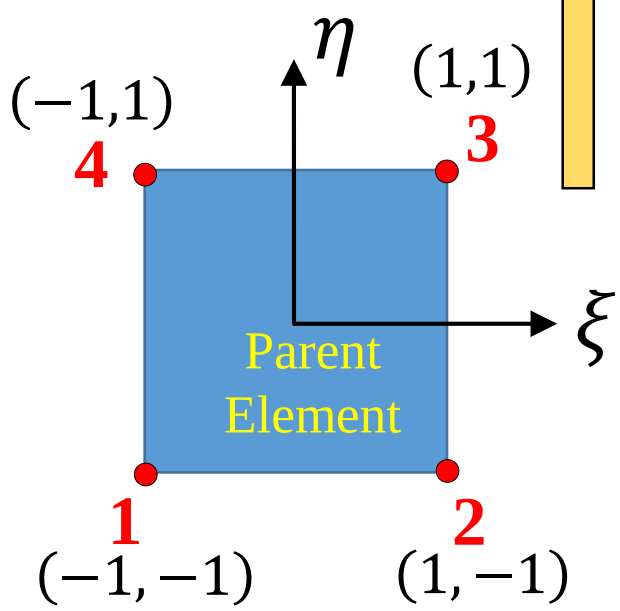
$n_{gp} = 2$

$$\Rightarrow [K^e] = k \sum_{i=1}^{n_{gp}} \sum_{j=1}^{n_{gp}} W_i W_j \left|\mathbf{J}\left(\xi_i, \eta_j\right)\right| \mathbf{B}^{e\mathrm{T}}(\xi_i, \eta_j) \, \mathbf{B}^e(\xi_i, \eta_j)$$

NPTEL

Step 2: Element Equation (Coefficient Matrix $[K^e]$)





$$\mathbf{B}^e(x,y) = \begin{bmatrix} \frac{\partial \phi_1^e(x,y)}{\partial x} & \frac{\partial \phi_2^e(x,y)}{\partial x} & \frac{\partial \phi_3^e(x,y)}{\partial x} & \frac{\partial \phi_4^e(x,y)}{\partial x} \\ \frac{\partial \phi_1^e(x,y)}{\partial y} & \frac{\partial \phi_2^e(x,y)}{\partial y} & \frac{\partial \phi_3^e(x,y)}{\partial y} & \frac{\partial \phi_4^e(x,y)}{\partial y} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [\mathrm{J}]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

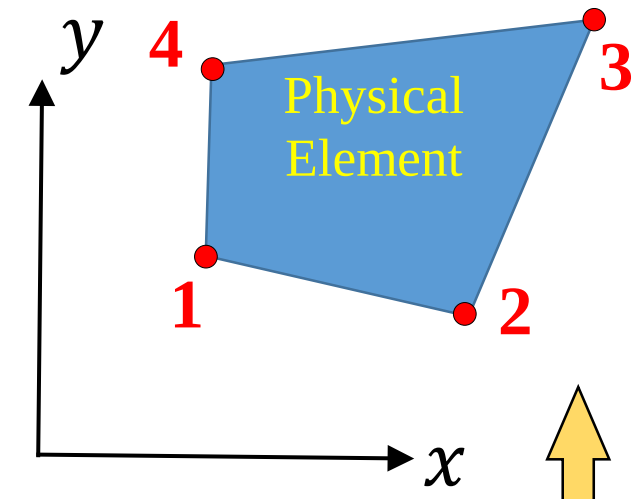
$$\mathbf{B}^e(\xi,\eta) = [\mathrm{J}]^{-1} \begin{bmatrix} \frac{\partial \phi_1}{\partial \xi} & \frac{\partial \phi_2}{\partial \xi} & \frac{\partial \phi_3}{\partial \xi} & \frac{\partial \phi_4}{\partial \xi} \\ \frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \frac{\partial \phi_3}{\partial \eta} & \frac{\partial \phi_4}{\partial \eta} \end{bmatrix}$$

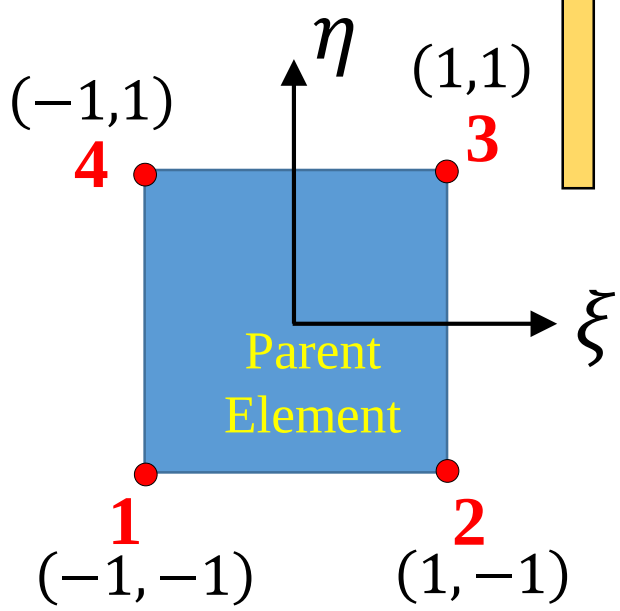
NPTEL

$$\phi_1(\xi,\eta) = (1-\xi)(1-\eta)/4 \qquad \phi_3(\xi,\eta) = (1+\xi)(1+\eta)/4$$

$$\phi_2(\xi,\eta) = (1+\xi)(1-\eta)/4 \qquad \phi_4(\xi,\eta) = (1-\xi)(1+\eta)/4$$

Step 2: Element Equation (Coefficient Matrix $[K^e]$)





$$\mathbf{B}^e(x,y) = \begin{bmatrix} \frac{\partial \phi_1^e(x,y)}{\partial x} & \frac{\partial \phi_2^e(x,y)}{\partial x} & \frac{\partial \phi_3^e(x,y)}{\partial x} & \frac{\partial \phi_4^e(x,y)}{\partial x} \\ \frac{\partial \phi_1^e(x,y)}{\partial y} & \frac{\partial \phi_2^e(x,y)}{\partial y} & \frac{\partial \phi_3^e(x,y)}{\partial y} & \frac{\partial \phi_4^e(x,y)}{\partial y} \end{bmatrix}$$

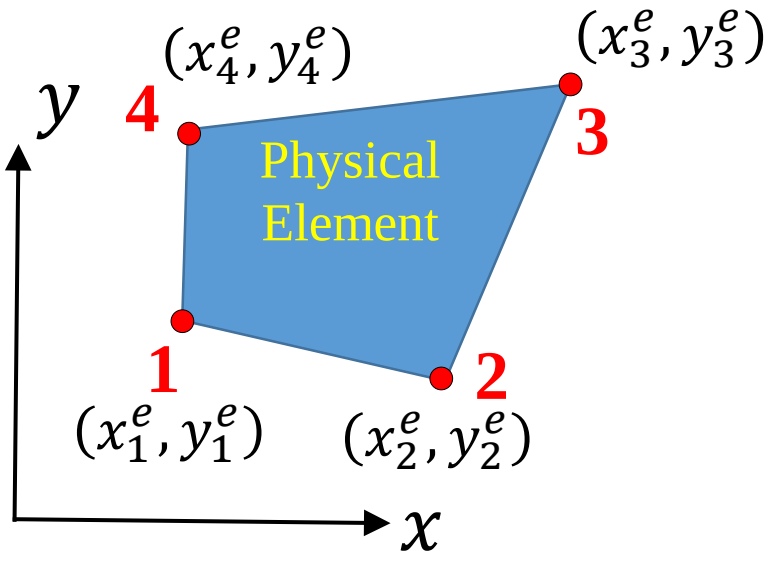
$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [\mathrm{J}]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

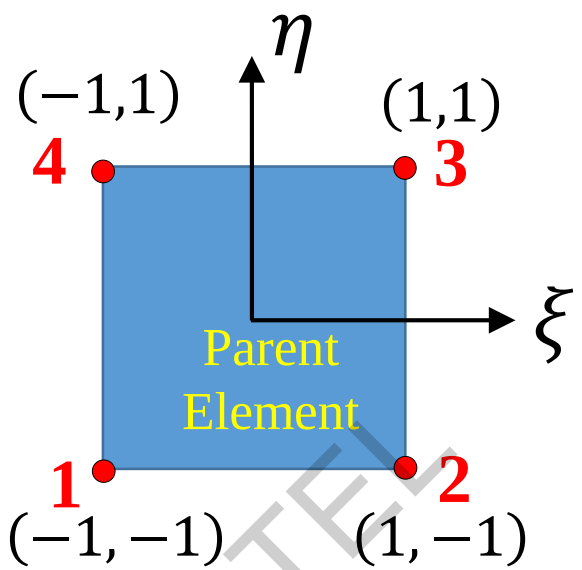
$$\mathbf{B}^e(\xi,\eta) = [\mathrm{J}]^{-1} \begin{bmatrix} \frac{\partial \phi_1}{\partial \xi} & \frac{\partial \phi_2}{\partial \xi} & \frac{\partial \phi_3}{\partial \xi} & \frac{\partial \phi_4}{\partial \xi} \\ \frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \frac{\partial \phi_3}{\partial \eta} & \frac{\partial \phi_4}{\partial \eta} \end{bmatrix}$$

NPTEL

$$\mathbf{B}^e(\xi,\eta) = [\mathrm{J}]^{-1} \frac{1}{4} \begin{bmatrix} \eta - 1 & 1 - \eta & 1 + \eta & -\eta - 1 \\ \xi - 1 & -\xi - 1 & 1 + \xi & 1 - \xi \end{bmatrix}$$

Step 2: Element Equation (Coefficient Matrix $[K^e]$)





$$[\mathrm{J}] = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} x_i^e & \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} y_i^e \\ \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} x_i^e & \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} y_i^e \end{bmatrix}$$

$$\phi_1(\xi, \eta) = (1-\xi)(1-\eta)/4$$

$$\phi_2(\xi, \eta) = (1+\xi)(1-\eta)/4$$

$$\phi_3(\xi, \eta) = (1+\xi)(1+\eta)/4$$

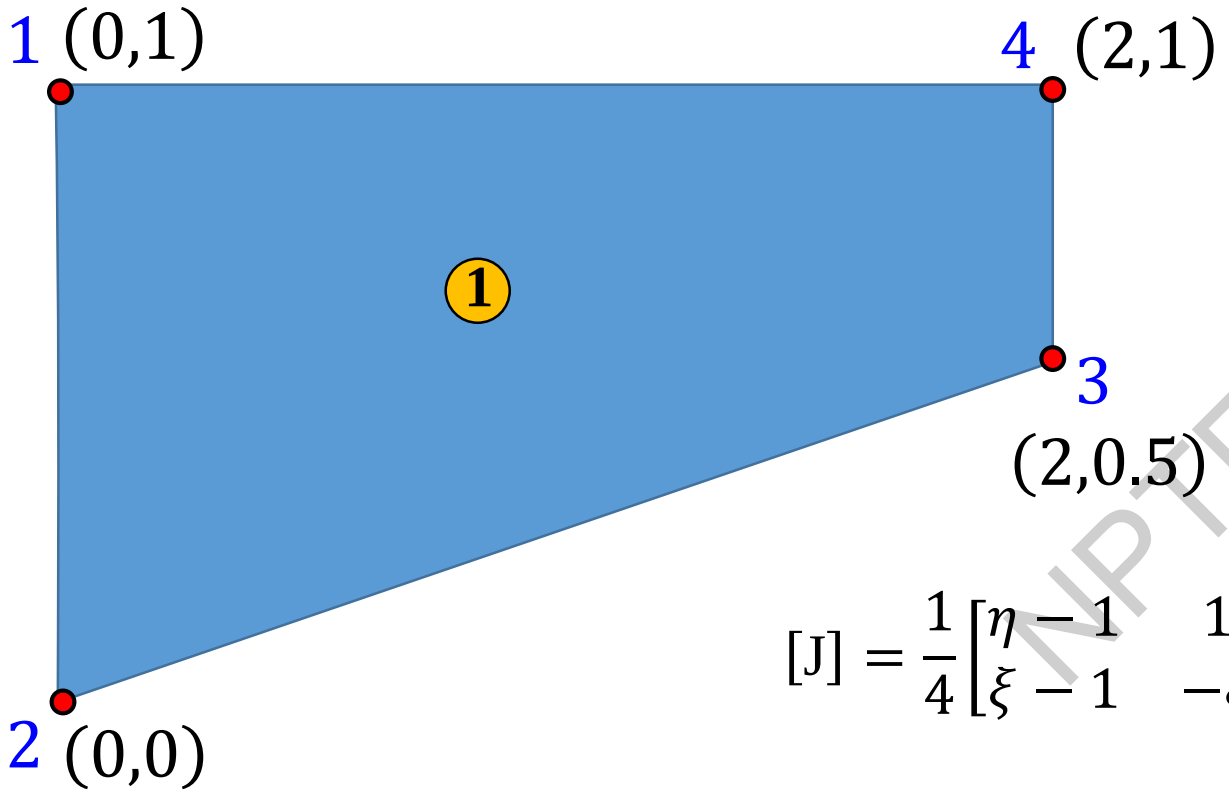
$$\phi_4(\xi, \eta) = (1-\xi)(1+\eta)/4$$

$$[\mathrm{J}] = \begin{bmatrix} \frac{\partial \phi_1}{\partial \xi} & \frac{\partial \phi_2}{\partial \xi} & \frac{\partial \phi_3}{\partial \xi} & \frac{\partial \phi_4}{\partial \xi} \\ \frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \frac{\partial \phi_3}{\partial \eta} & \frac{\partial \phi_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} \eta - 1 & 1 - \eta & 1 + \eta & -\eta - 1 \\ \xi - 1 & -\xi - 1 & 1 + \xi & 1 - \xi \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix}$$

NPTEL

Step 2: Element Equation (Coefficient Matrix $[K^e]$)



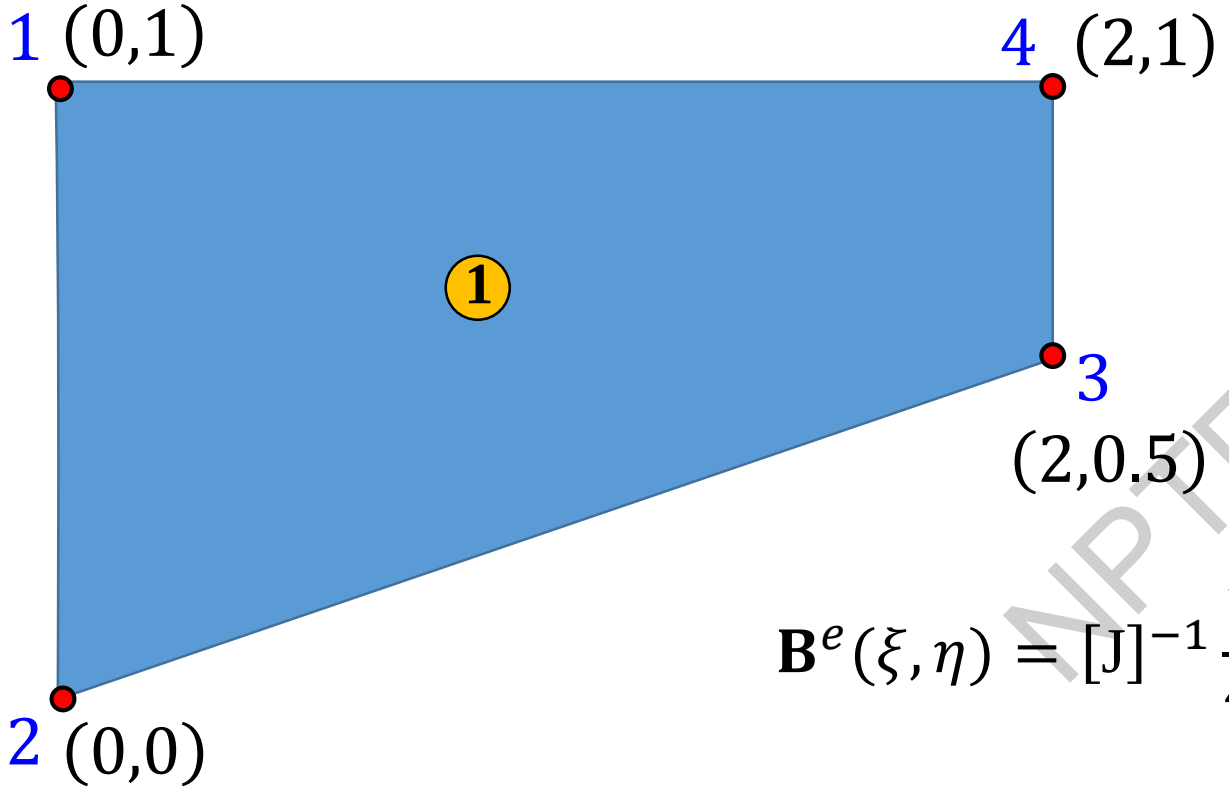
$$\begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0.5 \\ 2 & 1 \end{bmatrix}$$

NPTEL

$$[\mathrm{J}] = \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix}$$

$$\Rightarrow [\mathrm{J}] = \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0.5 \\ 2 & 1 \end{bmatrix} \Rightarrow [\mathrm{J}] = \begin{bmatrix} 0 & -0.375+0.125\eta \\ 1 & 0.125+0.125\xi \end{bmatrix}$$

Step 2: Element Equation (Coefficient Matrix $[K^e]$)



$$[\mathrm{J}] = \begin{bmatrix} 0 & -0.375 + 0.125\eta \\ 1 & 0.125 + 0.125\xi \end{bmatrix}$$

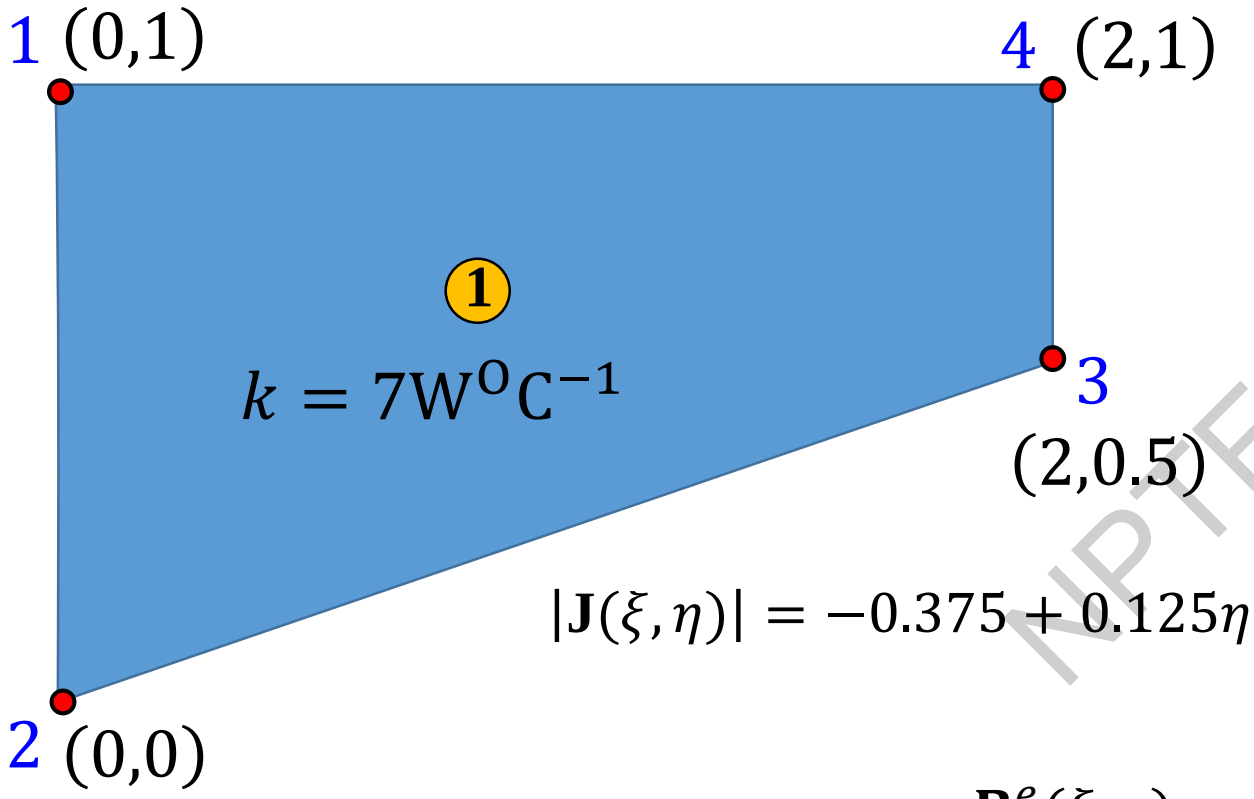
$$|\mathbf{J}| = -0.375 + 0.125\eta \qquad [\mathrm{J}]^{-1} = \begin{bmatrix} \dfrac{1+\xi}{3-\eta} & 1 \\ \dfrac{8}{\eta-3} & 0 \end{bmatrix}$$

$$\mathbf{B}^e(\xi,\eta) = [\mathrm{J}]^{-1}\frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}$$

$$\Rightarrow \mathbf{B}^e\ (\xi,\eta) = \frac{1}{4}\begin{bmatrix} \dfrac{1+\xi}{3-\eta} & 1 \\ \dfrac{8}{\eta-3} & 0 \end{bmatrix}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}$$

NPTEL

Step 2: Element Equation (Coefficient Matrix $[K^e]$)



$[i,j]$	ξ_i	η_j	$[W_i, W_j]$
[1,1]	$-1/\sqrt{3}$	$-1/\sqrt{3}$	[1,1]
[2,1]	$1/\sqrt{3}$	$-1/\sqrt{3}$	[1,1]
[1,2]	$-1/\sqrt{3}$	$1/\sqrt{3}$	[1,1]
[2,2]	$1/\sqrt{3}$	$1/\sqrt{3}$	[1,1]

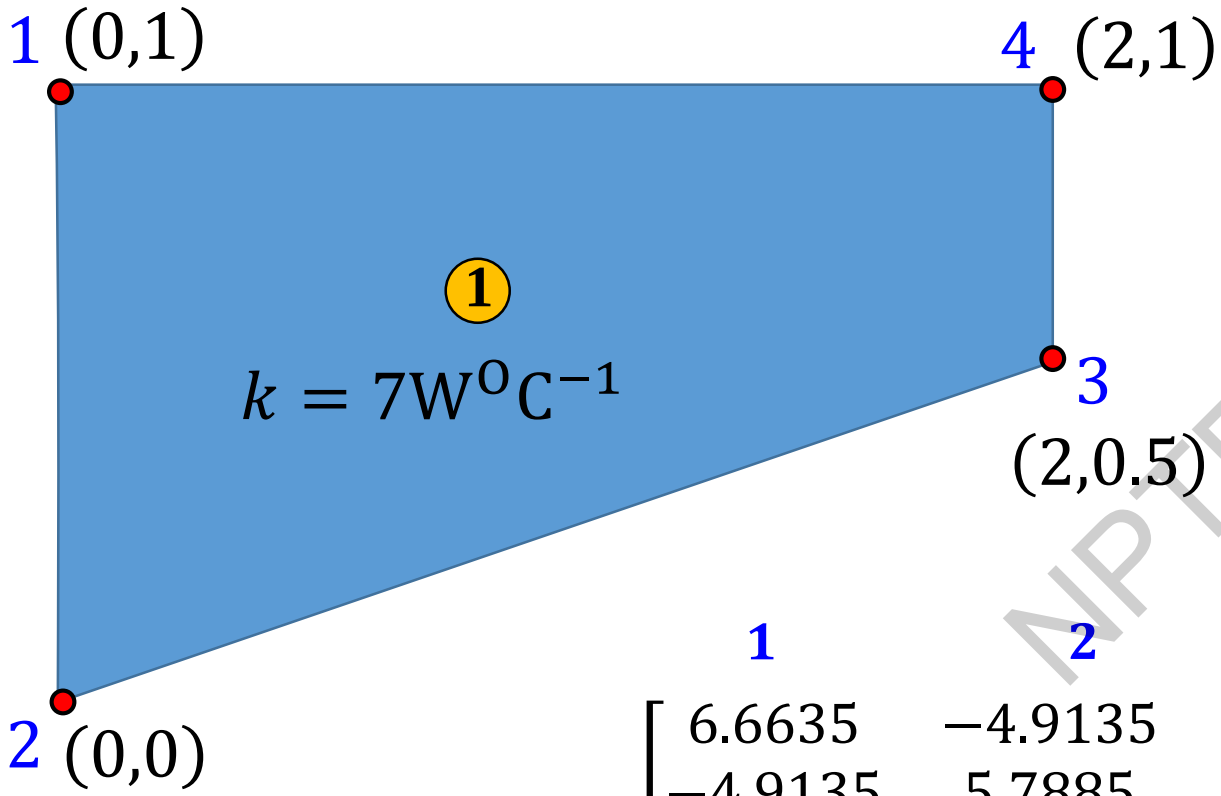
$$|\mathbf{J}(\xi,\eta)| = -0.375 + 0.125\eta$$

$$\mathbf{B}^e(\xi,\eta) = \frac{1}{4}\begin{bmatrix} \dfrac{1+\xi}{3-\eta} & 1 \\ \dfrac{8}{\eta-3} & 0 \end{bmatrix}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}$$

$$[K^e] = k\sum_{i=1}^{n_{gp}}\sum_{j=1}^{n_{gp}} W_i W_j \left|\mathbf{J}(\xi_i,\eta_j)\right| \mathbf{B}^{e\mathrm{T}}(\xi_i,\eta_j)\,\mathbf{B}^e(\xi_i,\eta_j)$$

NPTEL

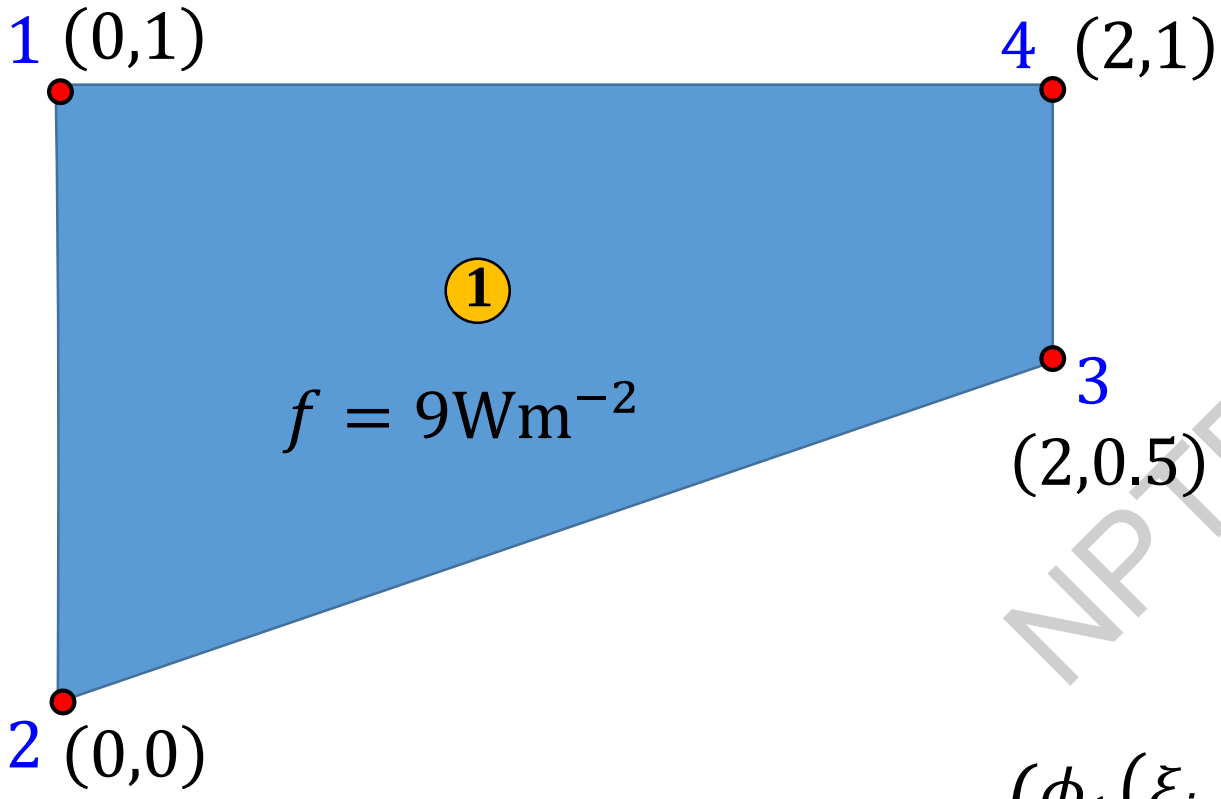
Step 2: Element Equation (Coefficient Matrix $[K^e]$)



NPTEL

$$K^e = \begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{matrix} \\ \begin{bmatrix} 6.6635 & -4.9135 & -4.1731 & 2.4231 \\ -4.9135 & 5.7885 & 2.4231 & -3.2981 \\ -4.1731 & 2.4231 & 9.1538 & -7.4038 \\ 2.4231 & -3.2981 & -7.4038 & 8.2788 \end{bmatrix} \end{array} \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{matrix}$$

Step 2: Element Equation (Source Term $\{f^e\}$)



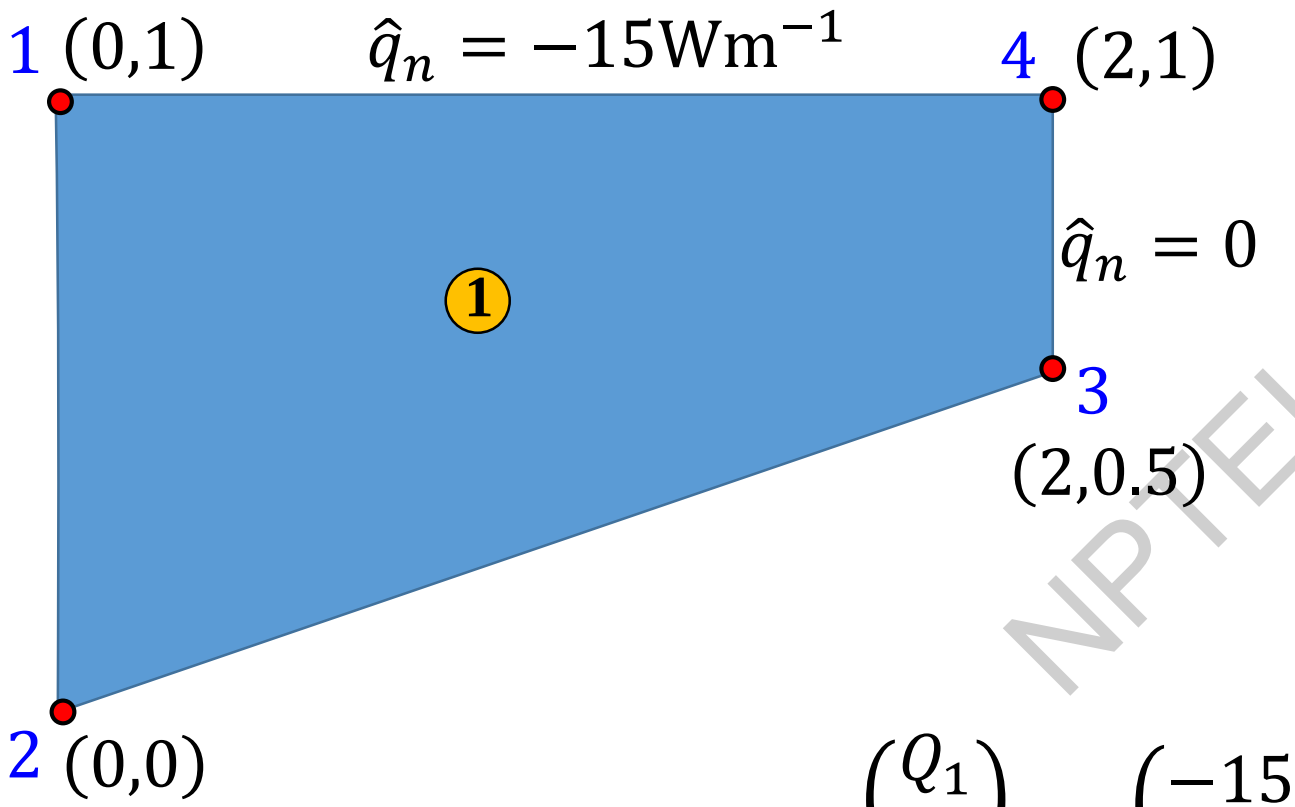
$$\{f^e\} = \int_{\Omega^e} f^e \phi_i^e \, dx dy$$

$[i,j]$	ξ_i	η_j	$[W_i, W_j]$
[1,1]	$-1/\sqrt{3}$	$-1/\sqrt{3}$	[1,1]
[2,1]	$1/\sqrt{3}$	$-1/\sqrt{3}$	[1,1]
[1,2]	$-1/\sqrt{3}$	$1/\sqrt{3}$	[1,1]
[2,2]	$1/\sqrt{3}$	$1/\sqrt{3}$	[1,1]

NPTEL

$$\{f^e\} = f \sum_{i=1}^{n_{gp}} \sum_{j=1}^{n_{gp}} W_i W_j \left|\mathbf{J}(\xi_i, \eta_j)\right| \begin{Bmatrix} \phi_1(\xi_i, \eta_j) \\ \phi_2(\xi_i, \eta_j) \\ \phi_3(\xi_i, \eta_j) \\ \phi_4(\xi_i, \eta_j) \end{Bmatrix} \Rightarrow \{f^e\} = \begin{Bmatrix} 3.75 \\ 3.75 \\ 3.00 \\ 3.00 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Step 2: Element Equation (Boundary Flux Term $\{Q^e\}$)

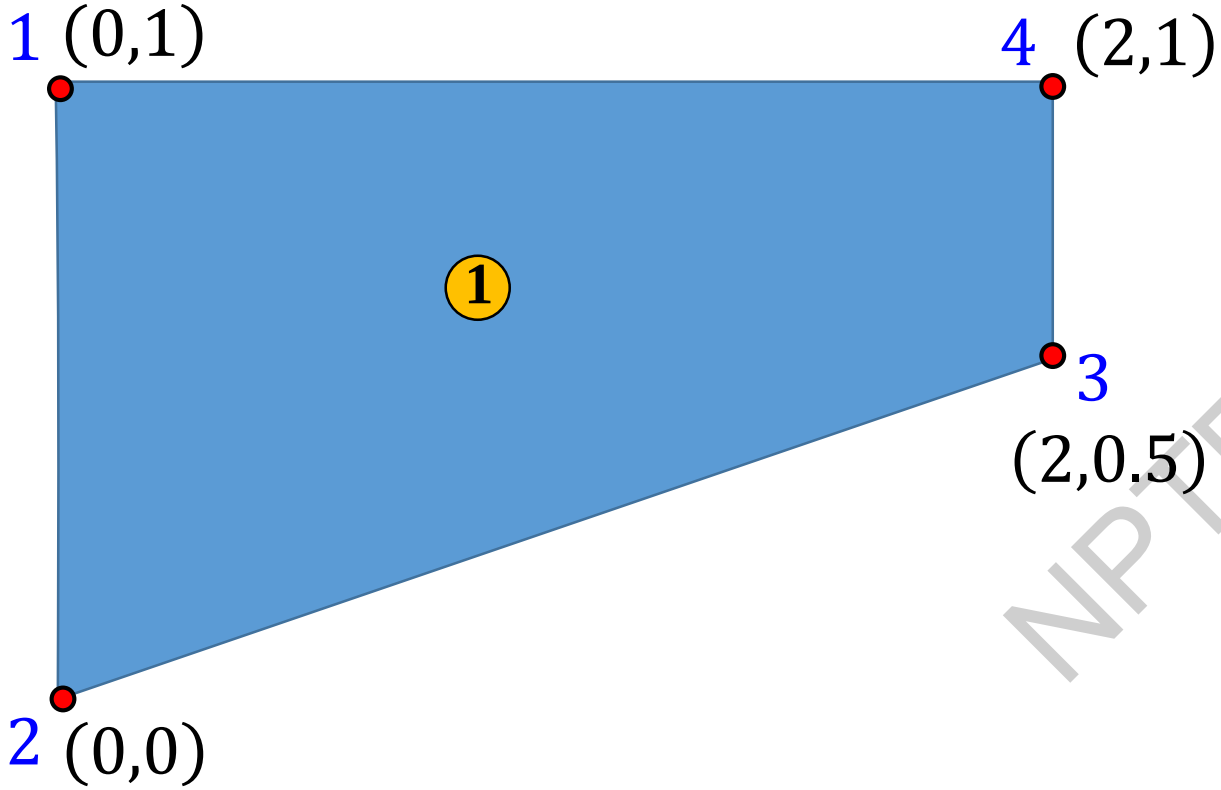


$$\{Q^e\} = \oint_{\Gamma^e} \phi_i^e q_n^e \, ds$$

NPTEL

$$\{Q^e\} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} -15 \\ 0 \\ 0 \\ -15 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

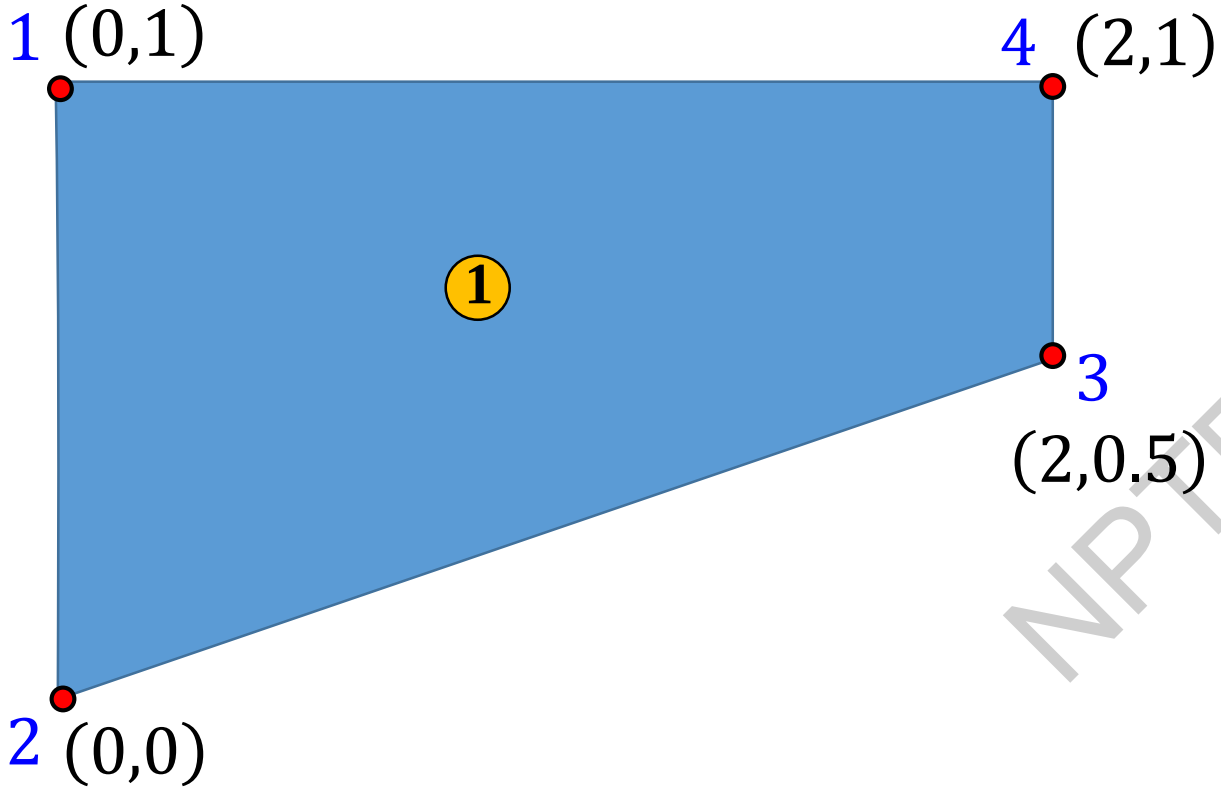
Step 3: Assembly of Element Equation



NPTEL

$$\begin{bmatrix} 6.6635 & -4.9135 & -4.1731 & 2.4231 \\ -4.9135 & 5.7885 & 2.4231 & -3.2981 \\ -4.1731 & 2.4231 & 9.1538 & -7.4038 \\ 2.4231 & -3.2981 & -7.4038 & 8.2788 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 3.75 \\ 3.75 \\ 3.00 \\ 3.00 \end{Bmatrix} + \begin{Bmatrix} -15 \\ 0 \\ 0 \\ -15 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \end{Bmatrix}$$

Step 4: Imposition of Boundary Conditions

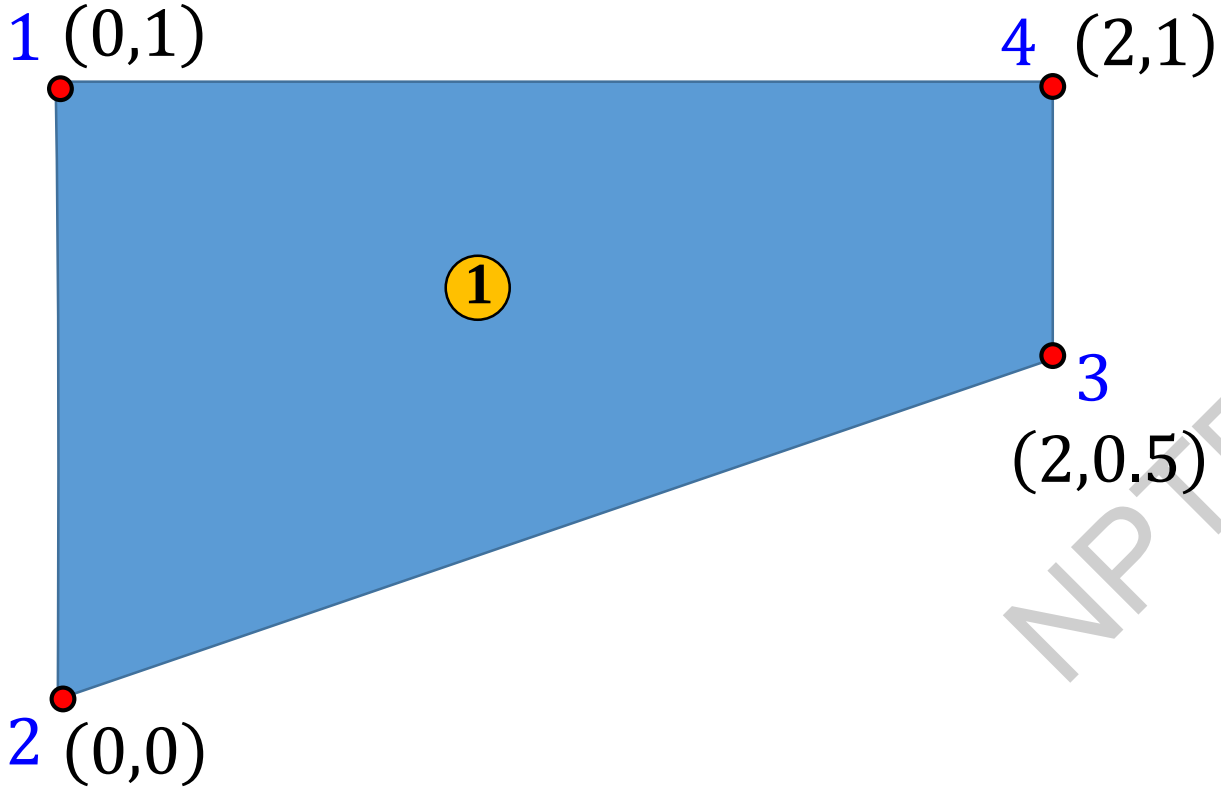


NPTEL

$$\begin{bmatrix} 6.6635 & -4.9135 & -4.1731 & 2.4231 \\ -4.9135 & 5.7885 & 2.4231 & -3.2981 \\ -4.1731 & 2.4231 & 9.1538 & -7.4038 \\ 2.4231 & -3.2981 & -7.4038 & 8.2788 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} -11.25 \\ 3.75 \\ 3.00 \\ -12 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \end{Bmatrix}$$

$T_1 = 0$, $T_2 = 0$, $T_3 = 0$

Step 5: Solutions

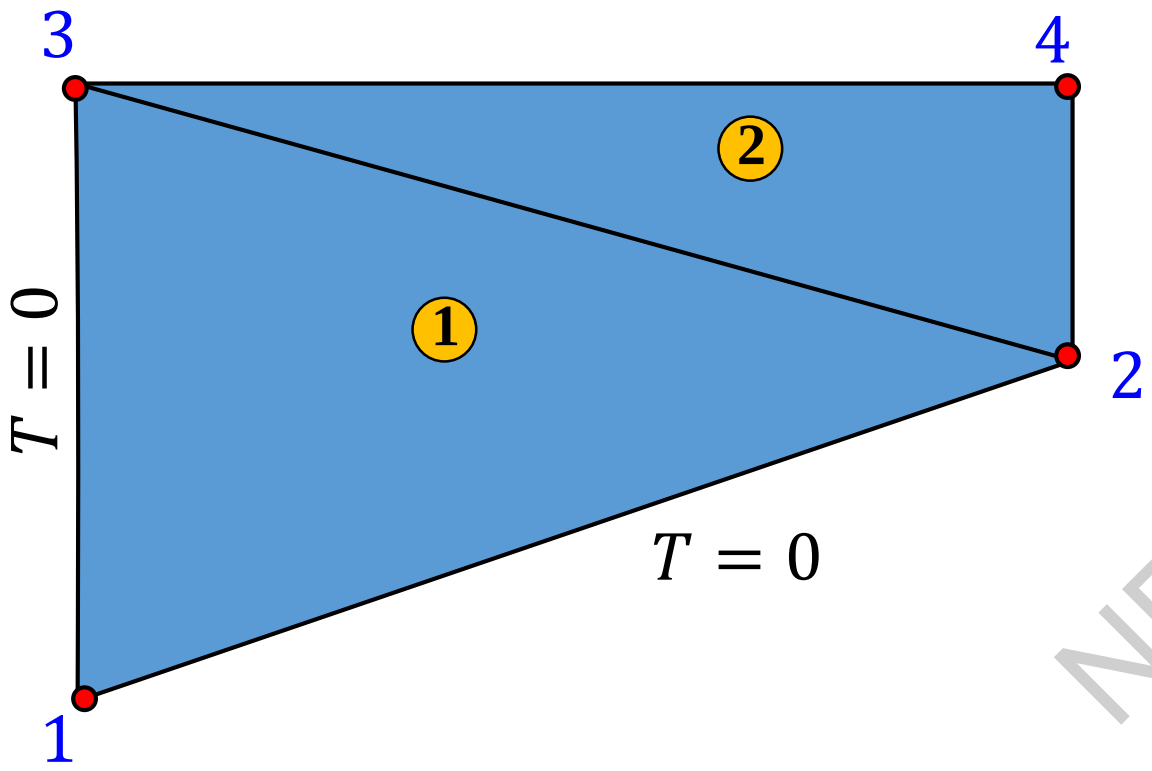


$$T_4 = -\frac{12}{8.2788} = -1.4495$$

NPTEL

$$\left[\begin{array}{cccc} 6.6635 & -4.9135 & -4.1731 & 2.4231 \\ -4.9135 & 5.7885 & 2.4231 & -3.2981 \\ -4.1731 & 2.4231 & 9.1538 & -7.4038 \\ \hline 2.4231 & -3.2981 & -7.4038 & 8.2788 \end{array}\right] \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} -11.25 \\ 3.75 \\ 3.00 \\ -12 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \end{Bmatrix}$$

Recall: Solution with CST



$$T_4 = -\frac{13.5}{14.875} = -0.90756$$

NPTEL

$$\left[\begin{array}{ccc:c} 7.4375 & -0.8750 & -6.5625 & 0 \\ -0.8750 & 15.7500 & -0.8750 & -14 \\ -6.5625 & -0.8750 & 8.3125 & -0.8750 \\ \hdashline 0 & -14 & -0.8750 & 14.8750 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4.5 \\ -11.5 \\ -13.5 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ 0 \end{Bmatrix}$$

Recall: Integration Over Quadrilateral Domain

$$\{f^e\} = \int_{\Omega^e} f^e \phi_i^e \, dx dy$$

$$\Rightarrow \{f^e\} = f \int_{\eta=-1}^{1} \int_{\xi=-1}^{1} |\mathbf{J}(\xi,\eta)| \begin{Bmatrix} \phi_1(\xi,\eta) \\ \phi_2(\xi,\eta) \\ \phi_3(\xi,\eta) \\ \phi_4(\xi,\eta) \end{Bmatrix} d\xi d\eta$$

$$\Rightarrow \{f^e\} = f \sum_{i=1}^{n_{gp}} \sum_{j=1}^{n_{gp}} W_i W_j \left|\mathbf{J}(\xi_i,\eta_j)\right| \begin{Bmatrix} \phi_1(\xi_i,\eta_j) \\ \phi_2(\xi_i,\eta_j) \\ \phi_3(\xi_i,\eta_j) \\ \phi_4(\xi_i,\eta_j) \end{Bmatrix}$$

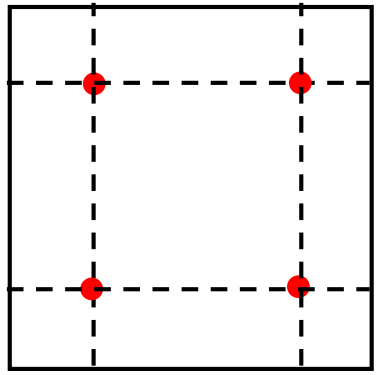
$$\phi_1(\xi,\eta) = (1-\xi)(1-\eta)/4$$

$$\phi_2(\xi,\eta) = (1+\xi)(1-\eta)/4$$

$$\phi_3(\xi,\eta) = (1+\xi)(1+\eta)/4$$

$$\phi_4(\xi,\eta) = (1-\xi)(1+\eta)/4$$

$n_{gp} = 2$



$[-1,1] \times [-1,1]$

NPTEL

Recall

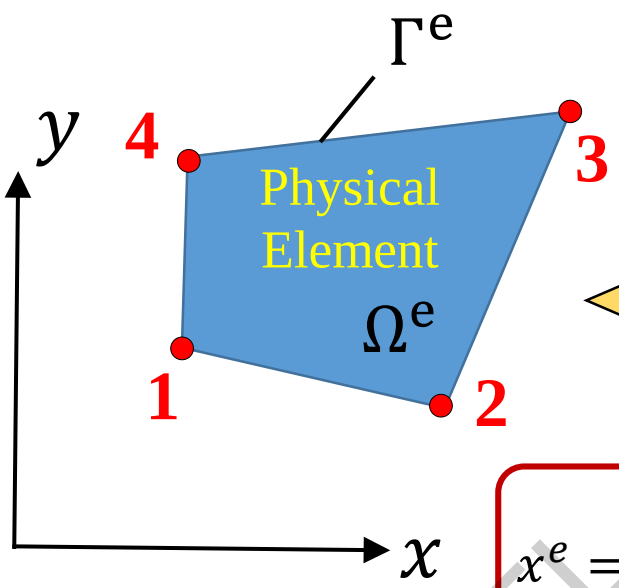
$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

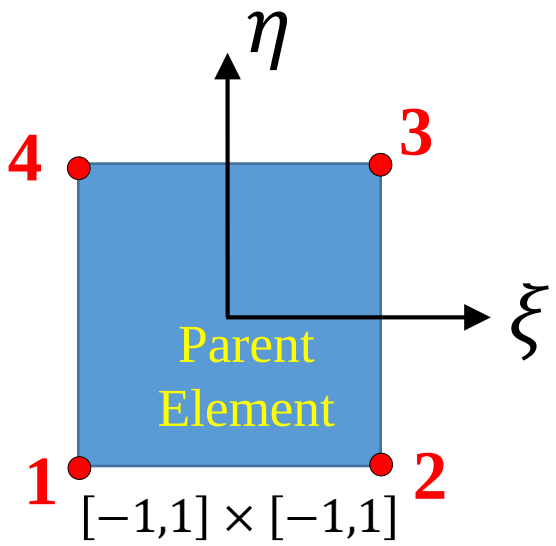
$$[K^e] = \int_{\Omega^e} \mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e dxdy$$

$$\{f^e\} = \int_{\Omega^e} f^e \phi_i^e dxdy$$

$$\mathbf{C} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\{Q^e\} = \oint_{\Gamma^e} \phi_i^e q_n^e \, ds$$





$$x^e = \sum_{i=1}^{4} \phi_i(\xi,\eta) x_i^e$$

$$y^e = \sum_{i=1}^{4} \phi_i(\xi,\eta) y_i^e$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [\mathrm{J}]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \phi_1^e(x,y)}{\partial x} & \frac{\partial \phi_2^e(x,y)}{\partial x} & \frac{\partial \phi_3^e(x,y)}{\partial x} & \frac{\partial \phi_4^e(x,y)}{\partial x} \\ \frac{\partial \phi_1^e(x,y)}{\partial y} & \frac{\partial \phi_2^e(x,y)}{\partial y} & \frac{\partial \phi_3^e(x,y)}{\partial y} & \frac{\partial \phi_4^e(x,y)}{\partial y} \end{bmatrix}$$

$$[\mathrm{J}] = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} x_i^e & \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} y_i^e \\ \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} x_i^e & \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} y_i^e \end{bmatrix}$$

NPTEL

NPTEL

THANK YOU !

NPTEL ONLINE CERTIFICATION COURSES

Finite Element Method

Week 8: Finite element formulation for 2D scalar field problems

Lecture 41: Computer Implementation

Biswanath Banerjee
Associate Professor
Department of Civil Engineering
IIT Kharagpur

Amit Shaw
Associate Professor
Department of Civil Engineering
IIT Kharagpur

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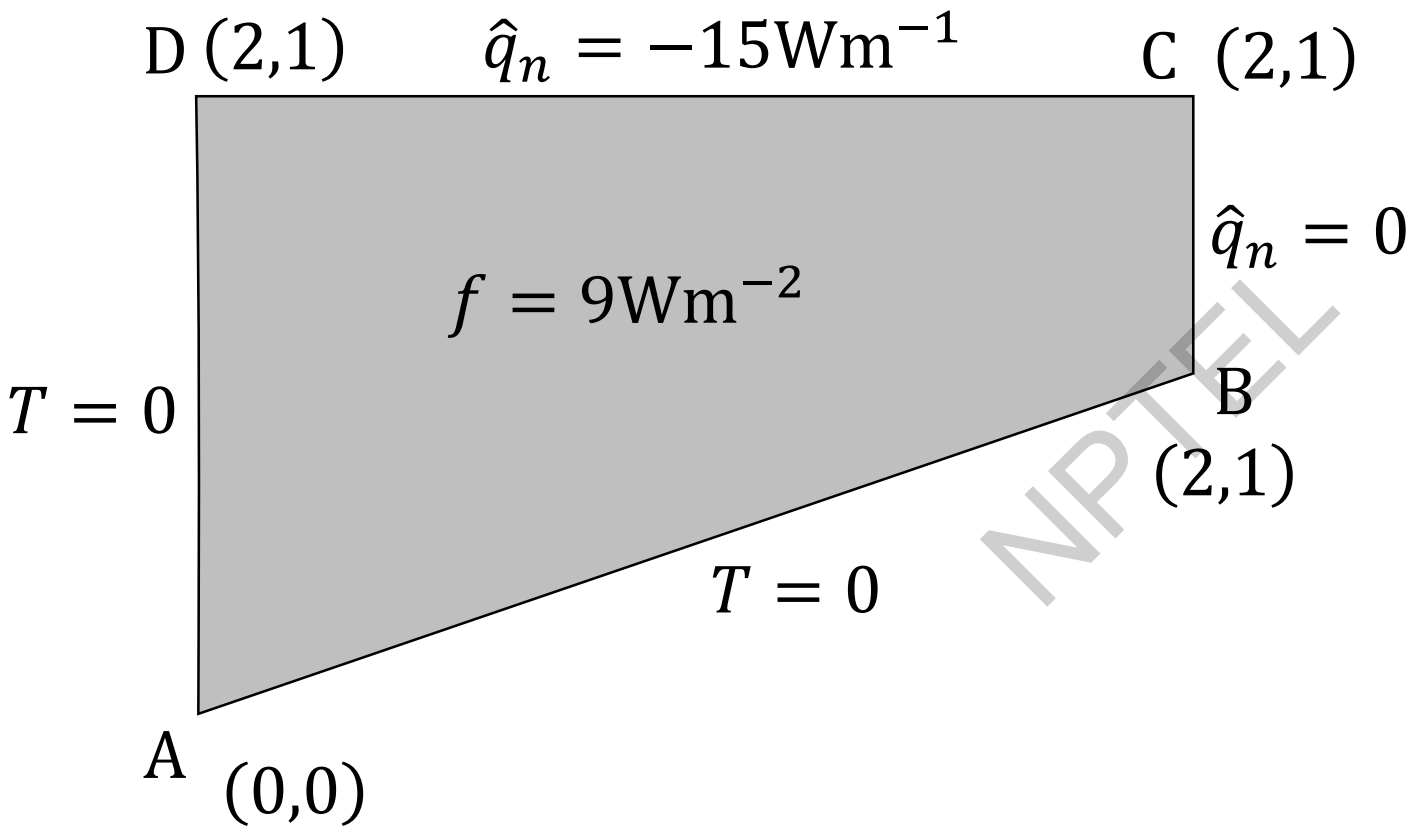
Recall: Heat Conduction Problem

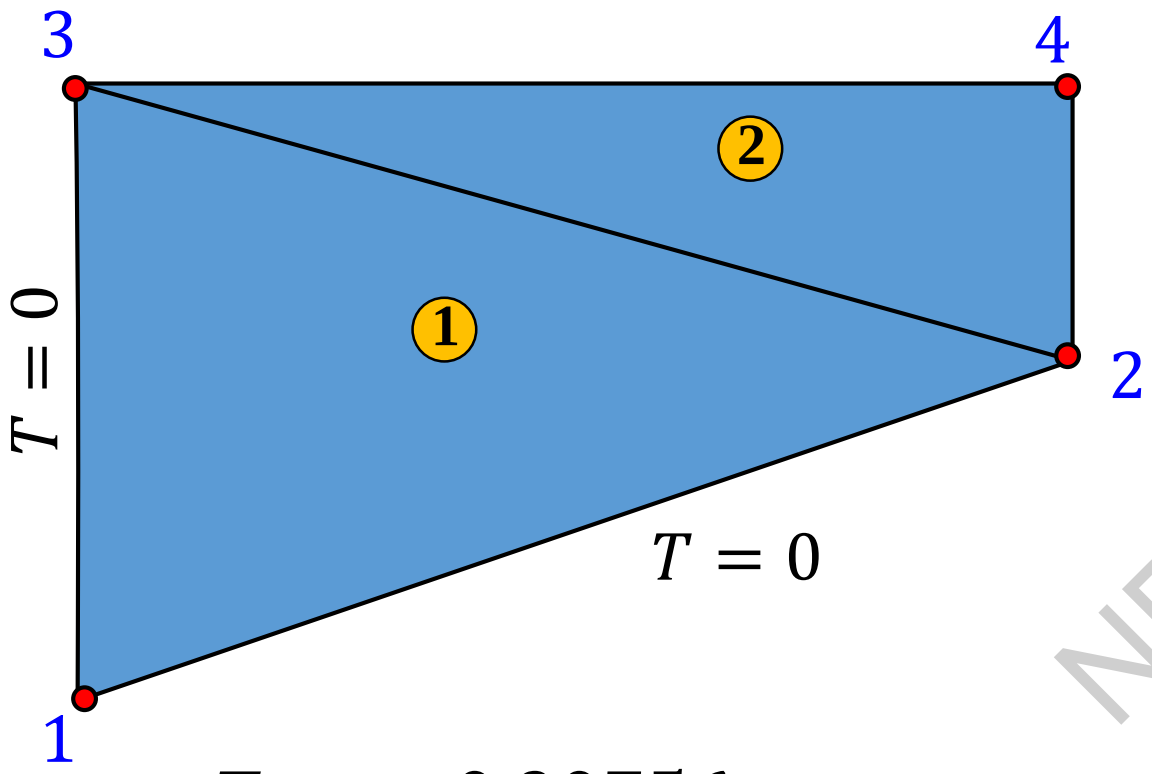
$$\nabla \cdot (k\nabla T) = -f$$



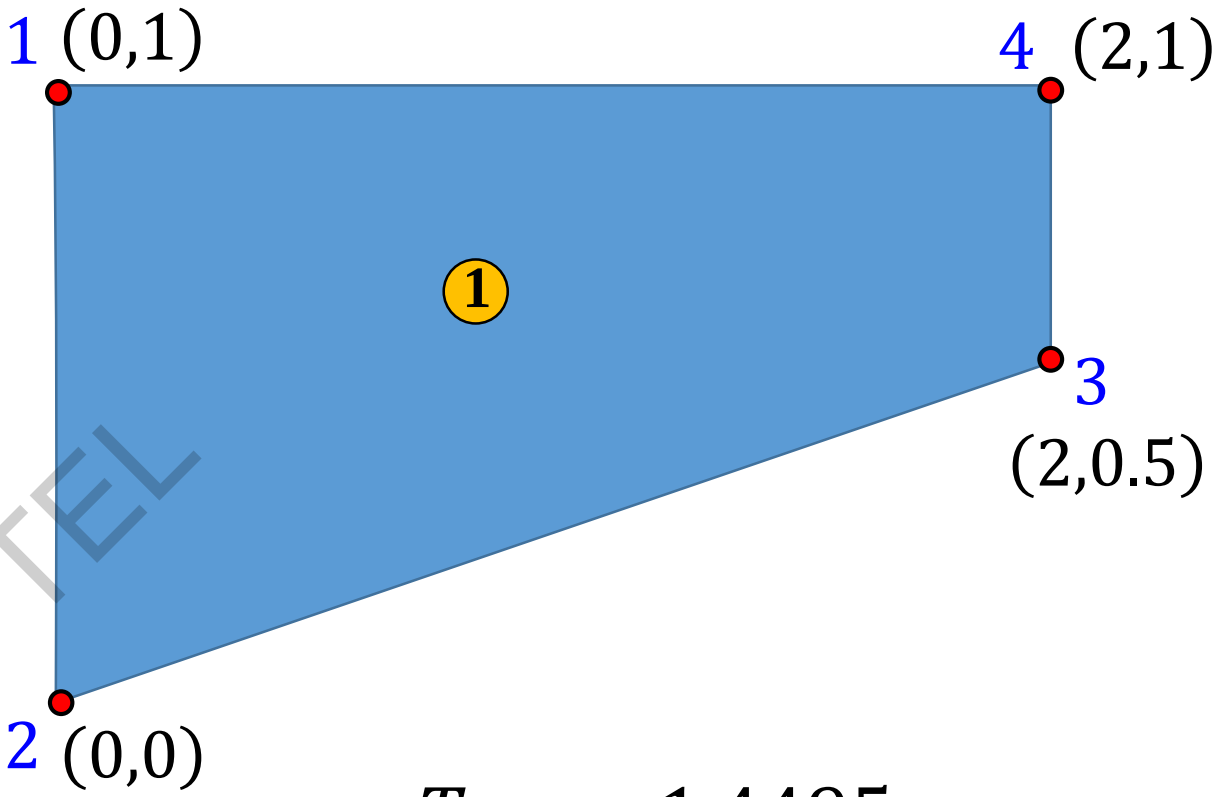
$$k = 7\mathrm{W}^{\mathrm{O}}\mathrm{C}^{-1}$$

NPTEL

Recall: Heat Conduction Problem



$T_4 = -0.90756$



$T_4 = -1.4495$

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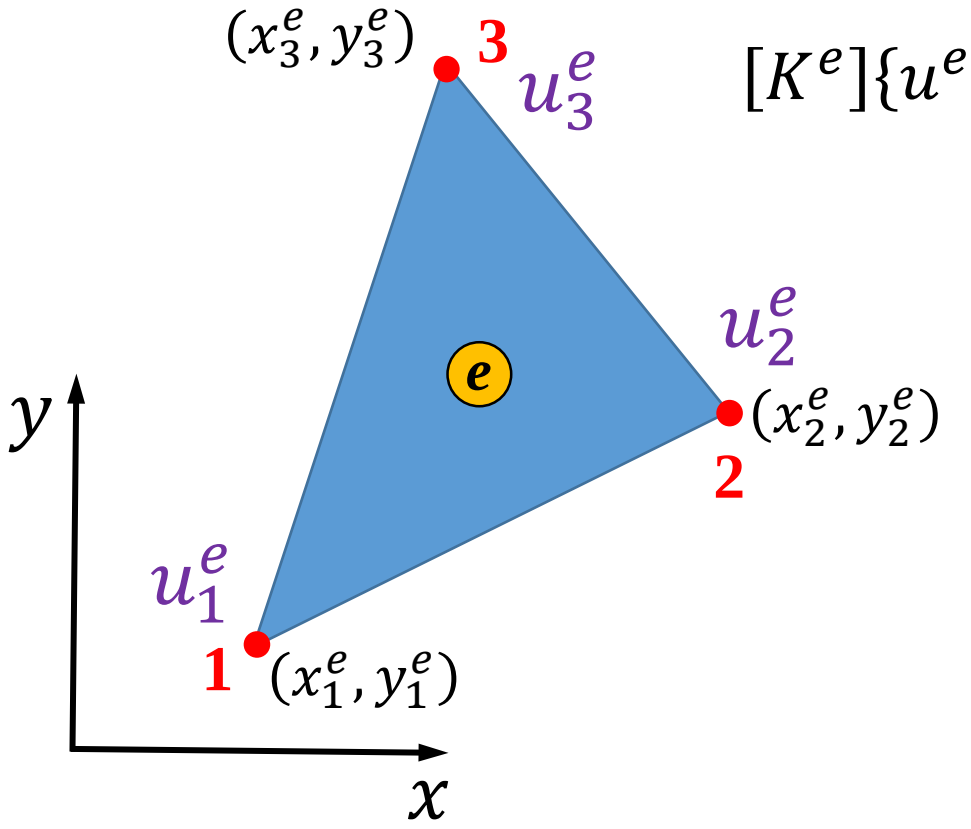
Recall:



$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\}$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e \, dxdy$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \frac{\partial \phi_3^e}{\partial x} \\ \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \frac{\partial \phi_3^e}{\partial y} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{B}^e = \frac{1}{2A^e} \begin{bmatrix} (y_2^e - y_3^e) & (y_3^e - y_1^e) & (y_1^e - y_2^e) \\ (x_3^e - x_2^e) & (x_1^e - x_3^e) & (x_2^e - x_1^e) \end{bmatrix}$$

Constant

Recall: Constant strain triangle

$$\mathbf{J} = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix}$$

$$2A^e = |\mathbf{J}|$$

$$[K^e] = \int_{\Omega^e} \mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e \, dxdy = \mathbf{B}^{e\mathrm{T}}\mathbf{C}\mathbf{B}^e \int_{\Omega^e} dxdy = A^e k \mathbf{B}^{e\mathrm{T}}\mathbf{B}^e$$

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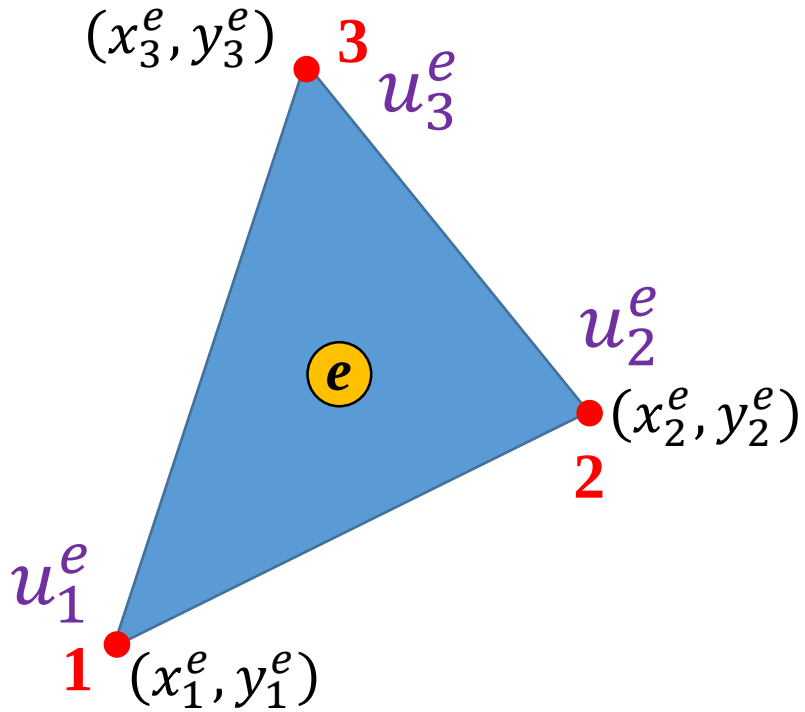
Recall:



$$\{f^e\} = \int_{\Omega^e} f \phi_i^e \, dxdy$$

For constant f

$$\{f^e\} = f \int_{\Omega^e} \phi_i^e \, dxdy = \frac{fA^e}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

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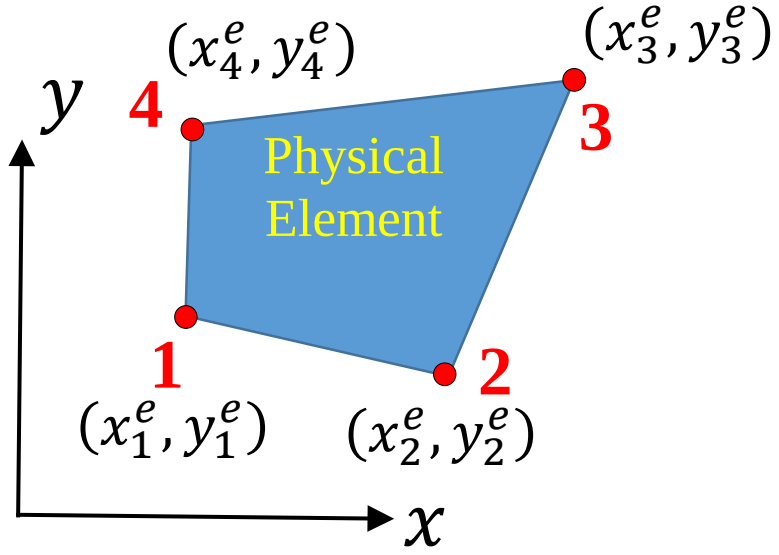
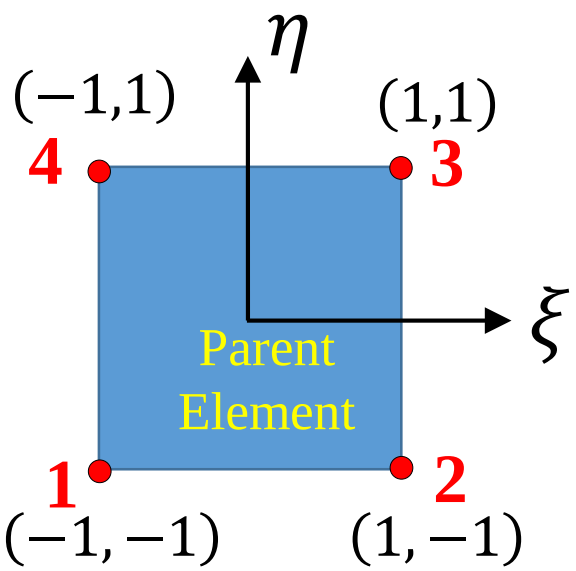
Recall:



$$[K^e] = k \sum_{i=1}^{n_{gp}} \sum_{j=1}^{n_{gp}} W_i W_j \left|\mathbf{J}(\xi_i, \eta_j)\right| \mathbf{B}^{e\mathrm{T}}(\xi_i, \eta_j)\, \mathbf{B}^e(\xi_i, \eta_j)$$



$$[\mathrm{J}] = \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix}$$

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$$\mathbf{B}^e(\xi, \eta) = [\mathrm{J}]^{-1} \frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}$$

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THANK YOU !